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UNIT -1

Fundamentals of calculus

In this chapter, we introduce the very important concepts of limits of continuity of the function, differentiation of non-trigourmetric functions. Basic applications of derivatives in finding marginal cost, marginal revenues etc. Increasing & decreasing function. Maxima & minima. Integration as reverse process of differentiation. Integration of some simple algebraic functions

1.1 Limit of a function : Let y=f(x) be a function of x and let 'a' be any real number we first understand what a 'limit' is. A limit is the value, a function approaches, as the independent variable of the function gets nearer and nearer to a particular value. In other words when x is very close to a certain number say a what is f (x) very close to ? It may be equal to f(a) but may be different. It may exist eve when f (a) is not defined

Meaning of $x \to$a

Let x be a variable and 'a' be a constant. It x assume values nearer and nearer to 'a' then we say that x tends to a or x approaches a and is written as '$x \to$a' by $x \to$a, we mean that $x \neq$ a and x may approach 'a' from left or right.

The expression the limit as x approaches to 'a' is written as $\lim_{x \to a}$ when x tends to 'a' from the left, this is called the left hand limit and is written as $\lim_{x \to a^-}$ Similarly as x decreases and approaches 'a' from 'right to left' this is called the 'right hand limit' and is written as $\lim_{x \to a^+}$

As $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) =$ l, we write that $\lim_{x \to a} f(x)$ =l we say that, $f(x)$ tends to the limit 'l' as x tends to 'a' The following are some simple algebric rules of limits.

1 $\lim_{x \to a} kf(x) = k \lim_{x \to a} f(x)$

2 $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

3 $\lim_{x \to a} f(x) \cdot g(x) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

4 $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ where $\lim_{x \to a} g(x) \neq o$

Note:

1 If the left hand limit of a function is not equal to the right hand limit of the function then the limit does not exist.

2 A limit equal to infinity does not imply that the limit does not exist.

Ex. The limit of a chord of the circle passing through a fixed point Q and a variable point

P, as P approaches Q is the tangent to the circle at P.

If PQ is a chord of a circle, when P approach Q along the circle, then the chord become the tangent to the circle at P.

While evaluating the limit we use the following methods.

1 Method of substitution

2 Method of factorization

3 Method of retionalization

4 Using the formula $\lim_{x \to a} \left[\frac{x^n - a^n}{x - a}\right] = na^{n-1}$

Method of substitution : In this method we directly substitute the value of x in the given function to obtain the limit value.

Ex.1 $\lim_{x \to 1} [x^2+4x+2] = (1)^2+4(1)+2 = 7$

2. $\lim_{x \to 3} \left[\frac{x^2+4x+1}{x+5}\right] = \frac{(3)^2+4(3)+1}{3+5} = \frac{22}{8} = \frac{11}{4}$

Indeterminate form : If $f(x) = \frac{x^2-9}{x-3}$ then $f(x) = \frac{0}{0}$ which is not defind this form is called an indeterminate form.

Method of factorization : If $\frac{f(x)}{g(x)}$ assumes an indeterminate form when $x = a$ then there exists a common factor for $f(x)$ and $g(x)$.

we remove the common factor and then use the substitution method to find the limit.

Ex. Evaluate $\lim_{x \to 2} \left[\frac{x^2-5x+6}{x^2-3x+2}\right]$

when $x=2$, $\frac{x^2-5x+6}{x^2-3x+2} = \frac{0}{0}$ which is an inderterminate form

Now $x^2-5x+6 = (x-3)(x-2)$

$x^2-3x+2 = (x-1)(x-2)$

$$\lim_{x \to 2} \left[\frac{x^2-5x+6}{x^2-3x+2}\right] = \lim_{x \to 2} \frac{(x-3)(x-2)}{(x-1)(x-2)}$$

$$= \lim_{x \to 2} \frac{x-3}{x-1}$$

$$= \frac{2-3}{2-1} = -1$$

Method of rationalization :

If $\frac{f(x)}{g(x)}$ assumes an indeterminate form when $x=a$ and $f(x)$ or $g(x)$ are irrational function then we rationalize $f(x)$, $g(x)$ and cancel the common factor, then using the substitution method to find the limit. Ex. Evaluate = $\lim_{x \to 0} \left[\frac{x}{1-\sqrt{1-x}}\right] \frac{0}{0}$

When x=0 the given function assumes the form which is an indeterminate form $\frac{0}{0}$ since g(x) the denominator is an irrational factor we multiply the numerators and denominator with the rationalizing factor of g(x)

$$\lim_{x \to 0} \left[\frac{x}{1-\sqrt{1-x}}\right] = \lim_{x \to 0} \frac{x}{(1-\sqrt{1-x})} \times \frac{1+\sqrt{1-x}}{1+\sqrt{1-x}}$$

$$= \lim_{x \to 0} \frac{x(1+\sqrt{1-x})}{1-1+x}$$

$$= \lim_{x \to 0} \frac{x(1+\sqrt{1-x})}{x}$$

$$= \lim_{x \to 0} (1+\sqrt{1-x})$$

$$= (1+\sqrt{1}) = 2$$

<u>Using the formula :</u> $\lim_{x \to a} \left[\frac{x^n-a^n}{x-a}\right] = na^{n-1}$ Where n is any rational number.

Ex. Evaluate

$$\lim_{x \to 4} \left[\frac{x^4-256}{x-4}\right]$$

$$= \lim_{x \to 4} \left[\frac{x^4-4^4}{x-4}\right]$$

$$= 4 \times 4^{4-1}$$

$$= 4 \times 4^3$$

$$= 256$$

Then find the limit of the function at x=0

clearly $\lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 = 0^2 = 0$

Thus $\lim_{x \to 0} f(x) = 0 = f(0)$

Hence, f is continuous at x=0

Ex3. Discuss the continuity of the function f given by f(x) = |x| at x=0

Solution : by definition

$$f(x) = \begin{cases} -x, \text{ it } x < 0 \\ x, \text{it } x \geq 0 \end{cases}$$

Clearly the function is defind at '0' and f(0) =0 left hand limit of f at '0' is

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-x) = 0$$

similarity, the right hand limit of f at '0' is

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x = 0$$

Thus $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} f(x) = f(0) = 0$

Hence f is continuous at x=0

Ex 4. Check the point where the constant function $f(x) = K$ is continuous.

Solution : The function is defined at all real numbers and by definition, its value at any real number equals K. let 'c' be any real number

Then $\lim_{x \to c} f(x) = \lim_{x \to c} K = K$

Since $f(c) = k = \lim_{x \to c} f(x)$ for any real number 'c' , the function f is continuous at every real number.

Ex 5. Prove that the identity function on real number given by $f(x) = x$ is continuons at every real number.

Solution : The function is clearly defined at every point and f (c) =c for every real number 'c' also. $\lim_{x \to c} f(x) = \lim_{x \to c} x = c$

Thus $\lim_{x \to c} f(x) = c = f(c)$ and hence the function is continouns at every real number.

Definition : A real function f is said to be continuous if it is continuous at every point in the domain of f.

Ex5. Is the function defined by $f(x) = |x|$, a continuos function?

Solution : We may rewrite as

$$f(x) = \begin{cases} -x, & \text{if } x<0 \\ x, & \text{if } x \geq 0 \end{cases}$$

we know that f is continuos at $x=0$

Let c be a real number such that c<0. Then f (c) = -c Also

$\lim_{x \to c} f(x) = \lim_{x \to c} (-x) = -c$

since $\lim_{x \to c} f(x) = f(c)$, f is continuos at all negative real numbers.

Now, let c be a real number such that c>0. Them f (c) = c Also

$\lim_{x \to c} f(x) = \lim_{x \to c} x = c$

since $\lim_{x \to c} f(x) = f(c)$, f is continuos at all positive real number hence f is continuos at all points.

Ex. Discuss the continuity of the function f defind by $f(x) = \frac{1}{x}$ $x \neq 0$

Solution: For any non zero real number c, we have $\lim_{x \to c} f(x) = \lim_{x \to c} \frac{1}{x} = \frac{1}{c}$

Also since for $c \neq 0$, $f(c) = \frac{1}{c}$, we have $\lim_{x \to c} f(x) = f(c)$ and hence, f is continouns at every point in the domain of f Thus f is continuous function.

Ex. Discusses the continuity of the function f defined

by $$f(x) = \begin{cases} x+2, & \text{if } x \leq 1 \\ x-2, & \text{if } x>1 \end{cases}$$

Solution : The function is defined at all points of the real liine.

Case 1. If c<1 Then f (c) = c+2 Therefore

$\lim_{x \to c} f(x) = \lim_{x \to c} (x+2) = c+2$

Thus, f is continuous at all real number less than 1

Case 2. If c>1, then f (c) = c-2 Therefore

$$\lim_{x \to c} f(x) = \lim_{x \to c} (x-2) = c-2 = f(c)$$

Thus, f is continuous at all point $x > 1$

Case 3. If c=1, then the left hand limit of f at $x=1$ is

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x+2) = 1+2 = 3$$

The right hand limit of f at $x=1$ is

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x-2) = 1-2 = -1$$

since the left and right hand limit of f at $x=1$ do not coincide. So f is not continuous at $x=1$. Hence $x=1$ is the only point of discontinuity of f.

Ex. Show that every polynomial function is continuous.

Solution : Let P is a polynomial function is defined by $P(x) = a_0 + a_1 x + \ldots a_n x^n$ for some natural number n, $a_n \neq 0$ and $a_i \in R$ clearly this function is defined for every real number for a fixed real number c we have $\lim_{x \to c} p(x) = p(c)$

by definition, P is continuous at c since c is any real number, P is continuous at every real number and hence Pis a continuous function.

Since continuity of a function at a point is entirely dictated by the limit of the function at that point it is reasonable to expect results analogous to the case of limits.

Suppose f and g be two real function continuous at a real number c then

1. f +g is continuous at $x=c$
2. f-g is continuous at $x=c$
3. f.g is continuous at $x=c$
4. $\left\{\frac{f}{g}\right\}$ is continuous at $x=c$ provided $g(c) \neq 0$

Ex. Prove that every rational function is continuous

Solution : Every rational function f is given by

$$f(x) = \frac{p(x)}{q(x)}, \quad q(x) \neq 0$$

Where p and q are polynomial function. The domain of f is all real number except point at which q is zero. Since polynomial function are continuous so by the property (4) of the above f is continuous.

Exercise 1

1. Find $\lim\limits_{x \to 1} \dfrac{\sqrt[5]{x}-1}{\sqrt[4]{x}-1}$ Ans. $\dfrac{4}{5}$

2. $\lim\limits_{x \to 0} \dfrac{|x|}{x}$ Ans. -1

3. Evaluate $\lim\limits_{x \to 3} \dfrac{|x-3|}{x-3}$ Ans. Limit doesn't exist

4. Evaluate $\lim\limits_{x \to 3} \left[\dfrac{x^3-27}{x-3}\right]$ Ans. 27

5. Evaluate $\lim\limits_{x \to 2} \left[\dfrac{2x^2-9x+10}{5x^2-5x-10}\right]$ Ans. $\dfrac{1}{15}$

6. Evaluate $\lim\limits_{x \to \infty} \dfrac{7x-3}{8x-10}$ Ans. $\dfrac{7}{8}$

7. Evaluate the continuity of the function

   $f(x)$ $2x^2-1$ at $x=3$

8. Examine the following function for continuity

   (a) $f(x) = x-5$ (b) $f(x) = \dfrac{1}{x-5}$, $x \neq 5$

   (c) $f(x) = \dfrac{x^2-25}{x+5}$ $x \neq -5$ (d) $f(x) = |x-5|$

9. Find all points of discontinuity of f, where f is defined by :-

   (a) $f(x) = \begin{cases} 2x+3, & \text{if } x \le 2 \\ 2x-3, & \text{if } x > 2 \end{cases}$

   (b) $f(x) = \begin{cases} |x|+3, & \text{if } x \le -3 \\ -2x, & \text{if } -3 < x < 3 \\ 6x+2, & \text{if } x \ge 3 \end{cases}$

   (c) $f(x) = \begin{cases} \dfrac{x}{|x|}, & \text{if } x < 0 \\ -1, & \text{if } x \ge 0 \end{cases}$

10. Discuss the continuity of the function f, where f is defined by

    $f(x) = \begin{cases} 3, & \text{if } 0 \le x \le 1 \\ 4, & \text{if } 1 < x < 3 \\ 5, & \text{if } 3 \le x \le 10 \end{cases}$

11. Find the relationship between a and b so that the function f defined by

    $f(x) = \begin{cases} ax+1, & \text{it } x \le 3 \\ bx+3, & \text{it } x > 3 \end{cases}$

    is continuous at $x=3$

12. Discuss the continuity of the following functions

    (a) $f(x) = \sin x + \cos x$ (b) $f(x) = \sin x - \cos x$

    (c) $f(x) = \sin x . \cos x$

13. Examine that Sin$|x|$ is a continuous function

14. Find all the points of discontinuity of f defined by $f(x) = |x| - |x+1|$

Derivatives : Suppose f is a real function and c is a point in its domain. The derivative of f at 'c' is defined by $\lim\limits_{h \to 0} \frac{f(c+h) - f(c)}{h}$ provided this limit exists derivative of f at c is denoted by f ' (c) or $\frac{d}{dx}(f(x))$ the function defined by $f'(x) = \lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h}$ wherever the limit exists is defined to be the derivative of f. The derivative of f is denoted by $f'(x)$ or $\frac{d}{dx}(f(x))$ or $\frac{dy}{dx}$ or y'. The process of finding derivative of a function is called differentiation finding the derivative of this way is called derivative from first principle.

The following rules were established as a part of algebra of derivatives.

1. $(u \pm v)' = u' \pm v'$
2. $(uv)' = u'v + uv'$ (Product rule)
3. $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$, wherever $v \neq 0$ (Quotient rule)

Note : Whenever we defined derivative, we had put a certain caution provided the limit exist. If $\lim\limits_{h \to 0} \frac{f(c+h) - f(c)}{h}$ doesn't exist we say that f is not differentiable at c we can also say that it both $\lim\limits_{h \to 0^-} \frac{f(c+h) - f(c)}{h}$ $\lim\limits_{h \to 0^+}$ and $\frac{f(c+h) - f(c)}{h}$ are finite and equal then f is differentiable at a point 'c'.

Ex. Find the derivative at $x=2$ of the function $f(x) = 3x$

Solution : We have

$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$
$$= \lim_{h \to 0} \frac{3(2+h) - 3(2)}{h}$$
$$= \lim_{h \to 0} \frac{6 + 3h - 6}{h}$$
$$= \lim_{h \to 0} \frac{3h}{h} = \lim_{h \to 0} 3 = 3$$

Ex. Find the derivative of $f(x) = 10x$

Solution : Since $f'(x) = \lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{10(x+h) - 10(x)}{h}$$
$$= \lim_{h \to 0} \frac{10h}{h} = \lim_{h \to 0} 10 = 10$$

Ex. Find the derivative of $f(x) = \frac{1}{x}$

Solution : We have

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{\frac{1}{(x+h)} - \frac{1}{x}}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{x - (x+h)}{x\,(x+h)}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{-h}{x\,(x+h)}\right] = \lim_{h \to 0} \frac{-1}{x\,(x+h)} = \frac{-1}{x^2}$$

Theorem : Derivative of $f(x) = x^h$ is hx^{h-1} for any positive integer h

Proof : $f'(x)$ $= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

with the application of binomial theorem

$$f'(x) = \lim_{h \to 0} \frac{h\,(nx^{n-1} + .. + h^{n-1})}{h}$$

$$= \lim_{h \to 0} (nx^{n-1} + .. + h^{n-1})$$

$$= nx^{n-1}$$

<u>Note</u> : The above theorem is true for all powers of x i.e., n can be any real number.

<u>Ex.</u> compute sol the derivative of $6x^{100} - x^{55} + x$

<u>Solution :</u>

$$f(x) = 6x^{100} - x^{55} + x$$

$$f'(x) = 100\ x^{99} - 55x^{54} + 1$$

<u>Ex.</u> Find the derivation of $f(x) = \frac{x+1}{x}$

<u>Solution</u> : Clearly the function is defined everywhere except at $x = 0$

$$\frac{d}{dx} f(x) = \frac{d}{dx} \frac{(x+1)}{x} = \frac{\frac{d}{dx}(x+1)\,x - (x+1)\frac{d}{dx}x}{x^2}$$

$$= \frac{1.x - (x+1)}{x^2}$$

$$= \frac{-1}{x^2}$$

<u>Exercise</u>

1. Find the derivative of x^2-2 at x=10
2. Find the derivative of the following function from first principle.

   (I) $x^3 - 27$ (ii) $(x-1)(x-2)$

   (iii) $\frac{1}{x^2}$ (vi) $\frac{x+1}{x-1}$

3. Find the derivative of $\frac{x^n - a^n}{x-a}$ for some constant a

4. Find the derivative of f from first principal $f(x) = x+\frac{1}{x}$

Basic application of derivatives in finding marginal cost, marginal revenues : Economics has differentiation tools like marginal cost and marginal revenues as its basic necessities from calculating the change in demand for a product to the its cost price to estimating the rate of change in its cost price to estimating the rate of change in revenue with an increase in selling price, every thing in practice can be efficiently found but by taking the derivative of the dependent variable of interest with respect to the independent variable.

The cost function : The total cost of producing of x number of products, represented by c (x) can be written as $c(x) = f(x) + v(x)$

Where $f(x)$ = The fixed cost, independent of the number of products being manufactured.

$v(x)$ = The variable cost, which depends on the number of product being manufactured.

The term marginal comes into play when we need to ascertain the increase in any dependent variable with a unit change of the independent variable.

If c (x) is the total cost of producing x units, then the change in the total cost if one additional unit needs to be produced at an output level of x units is given by.

Marginal cost = $\frac{d}{dx} c(x)$

The revenue and the profit functions: Revenue function R (x) represents the amount of money earned (the total turn over) by a company, by selling x number of product. If the selling price of every unit is equal to SP, the revenue function would be $R(x) = SP(x)$

The marginal Revenue : The rate of change of revenue per unit change in the output (number of products) is the marginal revenue given by $\frac{d}{dx}R(x)$

<u>Ex.</u> A firm is selling 100 units at a price of Rs. 250 however, to sell 110 units they need to cut the price down to Rs. 240 what is the level of marginal revenue at this higher level of sales ?

<u>Solution</u> : We can write the total revenue function for 100 units as - R (100) = 100x250 = Rs 25000 Similarly for 110 units - R (110) = 110 x 240 = Rs. 26400

Marginal Revenue = $\frac{26400-25000}{10}$

=Rs 140

Ex. The cost function for the manufacture of x number of goods by a company is

$c(x) = x^3 - 9x^2 + 24x$

Find the level of output at which the marginal cost is minimum.

Solution : we calculate the marginal cost as

$$\frac{d}{dx} c(x) = \frac{9}{dx} (x^3 - 9x^2 + 24x)$$

$$= 3x^2 - 18x + 24$$

In order to be a minimum at $x = x_0$ (say) it's derivative must vanish at x_0 thus

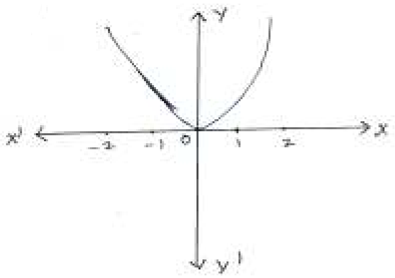
$$\frac{d}{dx}_{\; x = 0} (3x^2 - 18x + 24)$$

$$x_0 = 2,4$$

By the second derivative test, we can conclude that at x=4, the function assumes a minimum thus for an output = 4 finished goods, the marginal cost would be minimum.

Application of derivatives

Increasing and decreasing function, We will use differentiation to find out whether a function is increasing or decreasing or none. We can now illustrate with this example- consider the function f given by f $(x) = x^2, x \in$ R. The graph of this function is a parabola.



First consider the graph to the right of the origin when we move from left to right along the graph, the height of the graph continuously increases for this reason the function is said to be increasing for the real number x>0.

Now consider the graph to the left of the origin and observe here that as we move from left to right along the graph, the height of the graph continuously decrease consequently the function is said to be decreasing for the real number x<O.

Definition : Let I be an interval contained in the domain of a real valued function f. Then f is said to be.

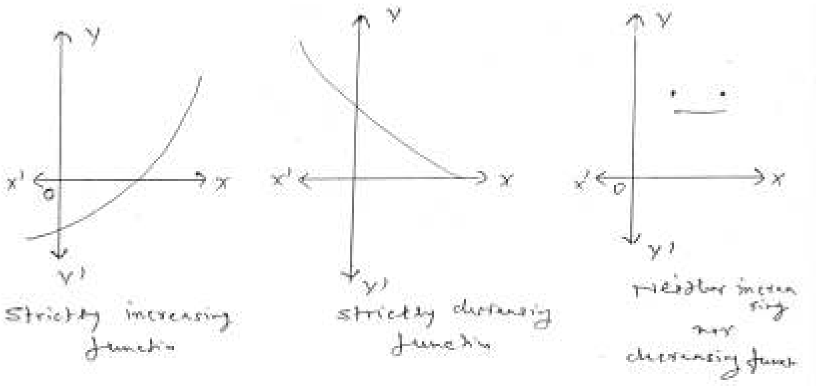
(i) increasing on I if $x_1 < x_2$ in I $\Rightarrow$ f (x_1) < f(x_2) for all $x_1, x_2 \in$ I

(ii) decreasing on I if $x_1 > x_2$ in I $\Rightarrow$ f (x_1) < f (x_2) for all x_1, $x_2 \in$ I

(iii) constant on I , if f(x) = c for all $x \in$ I where c is constant.

(iv) decreasing on I if $x, < x_2$ in I $\Rightarrow$ f $(x) \geq$ f (x_2) for all $x_1, x_2, \in$ I

(v) Strictly decreasing on I if $x_1 < x_2$ in I $\Rightarrow$ f $(x_1) >$, f (x_2) for all $x_{1,} x_2 \in$ I



we shall now define when a function is increasing or decreasing at a point.

Definition : Let x_0 be a point in the domain of definition of a real valued function f. Thus f is said to be increasing, decreasing at x_0 if there exist an open interval I containg x_0 such that f is increasing, decreasing respectively in I let us clearify this definition for the case of increasing function.

Ex. Show that the function given by

f $(x) = 7x$-3 is increasing in R

Solution : Let x_1 and x_2 be any two number in R. Thus

$x_1 < x_2 \Rightarrow 7x_1 < 7x_2 \Rightarrow 7x_1 - 3 < 7x_2 - 3$

$\Rightarrow f(x_1) < f(x_2)$

Thus f is strictly increasing on R.

Theorem : Let f be continuous on [a,b] and differentiable on the open interval (a,b) Then

(a) f is increasing in [a,b] if f' $(x) > 0$ for each $x \in$ (a,b)

(b) f is decreasing in [a,b] if f' $(x) < 0$ for each $x \in$ (a,b)

(c) f is a constant function in [a,b] if $f^1(x) = o$ for each $x \in$ (a,b)

Ex. Show that the function given by

$f(x) = x^3 - 3x^2 + 4x, \ x \in R$

is increasing on R.

Solution : Note that

$f^1(x) = 3x^2 - 6x, +4$

$= 3(x^2 - 2x + 1) + 1$

$= 3(x-1)^2 + 1 > o$, in every interval of R

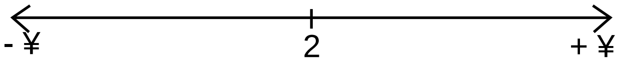
Therefore the function f is increasing on R.

Ex. Find the intervals in which the function f is given by $f(x) = x^2 - 4x + 6$ is

(a) increasing (b) decreasing

Solution :

$f(x) = x^2 - 4x + 6$



Exercise

1. Show that the function given by $f(x) = 3x + 17$ is increasing on R.
2. Find the intervals in which the function f given by $f(x) = 2x^2 - 3x$ is
   (a) Increasing (b) Decreasing
3. Find the intervals in which the following functions are strictly increasing or decreasing.
   (a) $x^2 + 2x - 5$ (b) $10 - 6x - 2x^2$ (c) $-2x^3 - 9x^2 - 12x+1$
4. Prove that the functions f given by $f(x)= x^2 - x + 1$ is neither strictly increasing nor decreasing on (-1,1).
5. Prove that the function given by $f(x) = x^3 - 3x^2 + 3x - 100$ is increasing in R.

Maxima and Minima : In this section, we will use the concept of derivative to calculate the maximum or minimum values of various functions. In fact, we will find the turning points of the graph of a function and thus find points at which the graph reaches its highest or lowest locally. The knowledge of such points is very useful in sketching the graph of a given function.

Definition : Let f be a function defined on an interval I. Then

(a) f is said to have a maximum value in I, if there exists a point C in I such that $f(C)> f(x)$ for all $x \in I$

The number $f(c)$ is called the maximum value of f in I and the point C is called a point of maximum value of f in I.

(b) f is said to have a minimum value in I, if there exist a point C in I such that $f(c) < f(x)$, for all $x \in I$

The number $f(c)$ in this case is called the minimum value of f in I and the point C in this case is called a point of minimum value f in I.

(c) f is said to have an extreme value in I if there exist a point C in I such that $f(c)$ is either a maximum value or a minimum value of f in I.

The number $f(c)$, in this case is called an extreme value of f in I and the point C is called an extreme point.

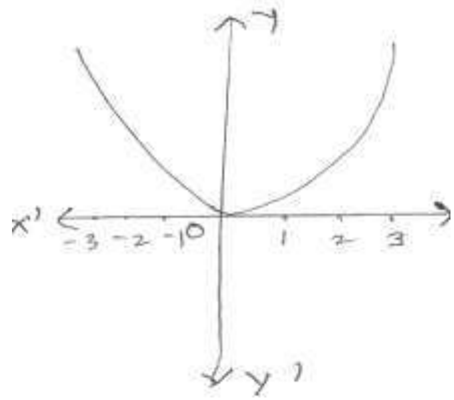
Ex. Find the maximum and the minimum values, if any, of the function f given by

$f(x) = x^2$, $x \in R$

Solution : From the graph of the give function we have

$f(x) = 0$ if $x = 0$ Also

$f(x) \geq 0$ for all $x \in R$



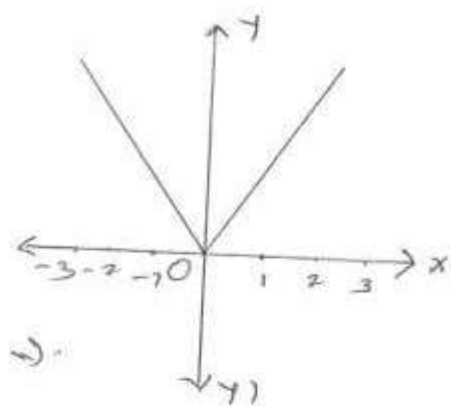
Therefore the minimum values of f is 0 and the point of minimum value of f is $x = 0$.

<u>Ex.:</u> Find the maximum and minimum values of f if any as the function given by

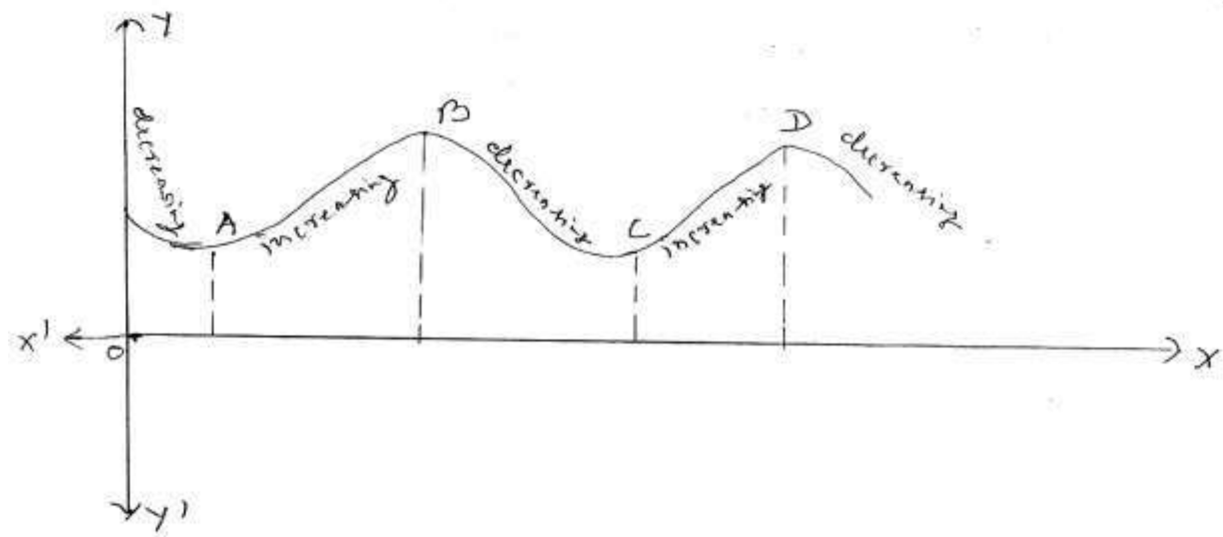
$f(x) = |x|$, $x \in R$.

<u>Solution :</u> From the graph of the function.

$f(x) \geq 0$ for all $x \in R$ and $f(x) = 0$ if $x = 0$



Therefore the function f has a minimum value 0 and the point of minimum value of f is x=0 Also the graph clearly shows that f has no maximum value in R and have no point of maximum value in R.



Let us now examine the graph of a function observe that at points A,B,C and D on the graph, the function changes its nature from decreasing to increasing or vice-versa. These points may be called turning points of the given function. Further observe that at turning points, the graph has either a little hill or a little valley. Roughly, the function has minimum value in some neighbourhood of each of the points A and C which are at the bottom of their respecting valley. Similarly the function has maximum value in some neighbourhood of point B and D which are at the top of their respective hills. For this reason. The point A and C may be regarded as points of local minimum value and points B and D may be regarded as points of local maximum for the function. The local maximum value and local minimum value of the function are referred to as local maxima and local minima resputinely of the function.

Theorem (without Proof): Let f be a function defined on an open interval I. suppose $C \in I$ be any point. If f has a local maxima or local minima at $x = C$, the neither $f'(C) = 0$ or f is not differentiable at C.

The converse of above theorem need not be true, that is, at a point at which the derivative vanished need not be a point of local maxima or local minima.

We shall now give a working rule for finding points of local maxima or points of local minima using only the first order derivatives.

Theorem (First derivative Test): Let f be a function defined on an open interval I. Let f be continues at a critical points C in I. Then

(1) If $f'(x)$ changes sign from positive to negative as x increase through C
i.$e.$, if $f'(x) > 0$ at every point sufficiently close to and to the left of C, and $f'(x)<0$ at every point sufficiently close to and to the right of C, then C is a point of local maxima.

(2) If $f'(x)$ changes sign from negative to positive as x increase through C
i.$e.$, if $f'(x) < 0$ at every point sufficiently close to and to the left of C, and $f'(x) > 0$ at every point sufficiently close to and to the right of C, then C is a point of local minima.

(3) If $f'(x)$ does not change sign as x increases through C then C is neither a point of local maxima nor a point of local minima. Infact such a point is called point of inflection.

Ex.: Find all points of local maxima and local minima of the function f given by

$$f(x) = x^3 - 3x + 3$$

Solution : We have

$$f(x) = x^3 - 3x + 3$$

Or $\quad f'(x) = 3x^2 - 3 = 3(x-1)(x+1)$

Or $\quad f'(x) = 0$ at $x = 1$ and $x = -1$

This $x = \pm 1$ are the only critical points which could possibly be the points of local maxima and/or local minima of f. Let us first examine the point $x = 1$

Note the value close to I and to the right of l, $f(x) > 0$ and for values close to I and to the left of, $f'(x) < 0$. Therefore by first derivative test $x = 1$ is a point of local minima and local minimum value is $f(1) = 1$.

In the case of $x = -1$ note that $f'(x) > 0$ for values close to and to the left of -1 and $f'(x) < 0$ for the values close to and to the right of -1

Therefore by first derivative test $x = -1$ is a point of local maxima and local maximum value is $f(-1) = 5$

Ex.: Find all the points of local maxima and local minima of the function f given by

$$(x) = 2x^3 - 6x^2 + 6x + 5$$

Solution : We have

$(x) = 2x^3 - 6x^2 + 6x + 5$

Or $f'(x) = 6x^2 - 12x + 6$

Or $f'(x) = 6\,(x^2 - 2x + 1)$

Or $f'(x) = 6\,(x - 1)^2$

Or $f'(x) = 0 \quad x = 1$

This $x = 1$ is the only critical point of f. we shall now examine this point for local maxima and or local minima of f. observe that $f'(x) \geq 0$ for all $x \in R$ and in particular $f'(x)>0$, for values close to I and to the left and to the right of 1. Therefore by first derivative test, the point $x = 1$ is neither a point of local maxima nor a point of local minima.

Hence $x = 1$ is a point of inflexion.

We shall now give another test to examine local maxima and local minima of a given function. This test is offer easier to apply than the first derivation test.

Theorem (Second derivative Test): Let f be a function defined on an interval I and $C \in I$. Let f be twice differentiable at C. Then.

(i) $x = c$ is a point of local maxima if $f'(c) = 0$ and $f''(c) < 0$. The value is local maximum value of f.

(ii) $x = c$ is a point of local minima if $f'(c) = 0$ and $f''(c) > 0$. In this case $f(c)$ is local minimum value of f.

(iii) The test fails if $f'(c) = 0$ and $f''(c) = 0$. In this case, we go back to the first derivative test and find whether C is a point of local maxima, local minima or a point of inflextion.

Ex.: Find local maxima and local minimum values of the function f given by

$$f(x) = 3x^4 + 4x^3 - 12x^2 + 12$$

Solution : We have

$f\ (x) = 3x^4 + 4x^3 - 12x^2 + 12$

Or $f'(x) = 12x^3 + 12x^2 - 24x$

$= 12x\ (x - 1)\ (x + 2)$

Or $f'(x) = 0$ at $x = 0$, $x = 1\ and\ x = -2$

Now $f\ ''(x) = 36x^2 + 24\ x - 24$

$= 12\ (3x^2 + 2x - 2)$

Or $f''(0) = -24 < 0$

$f''(1) = 36 > 0$

$f''(-2) = 72 > 0$

Therefore, by second derivative test, $x = 0$ is a point of local maxima and local maximum value of $f\ at\ x$=0 is $f(0)$= 12 while $x = 1$ and $x = -2$ are the points of local minima and local minimum value of $f\ at\ x = -1$ and -2 are $f(1) >$ and $f(-2)$= -20 respectively.

Ex.: Find all the points of local maxima and local minima of the function f given by

$$f\ (x) = 2x^3 - 6x^2 + 6x + 5$$

Solution : We have

$(x) = 2x^3 - 6x^2 + 6x + 5$

$f'(x) = 6x^2 - 12x + 6$

$= 6\ (x^2 - 2x + 1)$

$= 6\ (x - 1)^2$

$f''(x) = 12\ (x - 1)$

Now $f'(x) = 0$ gives $x = 1$. also $f''(1)$= 0 Therefore the second derivative test fails in this case. So we shall go back to the first derivative test. From first derivative test x=1 is neither a point of local maxima nor a point of local minima and so it is a point of inflexion.

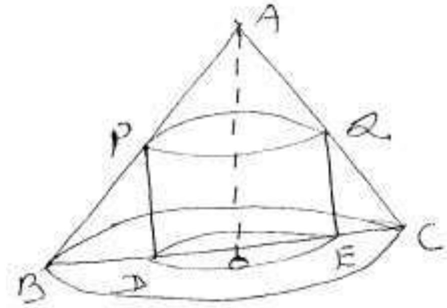
Ex.: Prove that the radius of the right circular cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone.

Solution : Let $oc = r$ be the radius of the cone and $OA = \mathrm{h}$ be its height. Let a cylinder with radius $OE = x$ inscribed in the given cone. The height QE of the cylinder is given by

$$\frac{QE}{OA} = \frac{EC}{OC} \qquad \therefore \Delta\, Q\, E\, C \sim \Delta\, A\, O\, C$$

Or $$\frac{QE}{\mathrm{h}} = \frac{r-x}{r}$$

Or $$QE = \frac{\mathrm{h}\,(r-x)}{r}$$



Let S be the curved surface area of the given cylinder. Then

$$S = S(x) = \frac{2\pi x h\,(r-x)}{r}$$

$$= \frac{2\,\pi\,h}{r}\,(rx - x^2)$$

Or $$S\,(x) = \frac{2\pi h}{r}\,(r - 2x)$$

$$S\,(x) = \frac{-4\pi h}{r}$$

Sina $S''\,(x) < 0\ for\ all\ x, S''\left(\frac{r}{2}\right) < 0$

So $x = \frac{r}{2}$ is a point of maxima of S. Hence, the radius of the cylinder of greatest curved surface area which can be inscribed in a given cone is half of the cone.

Exercise

1. Find the maximum and minimum value if any of the following functions given by.
   (i) $f(x)= (2x - 1)^2 + 3$ (ii) $f(x) = 9x^2 + 12x + 2$
   (iii) $f(x)= (x - 1)^2 + 10$ (iv) $g(x) = x^3 + 1$
2. Find the local maxima and local minima, if any of the following functions. Find also the local maximum and the local minimum value, as the case may me.
   (i) $f(x) = x^2$ (ii) $g(x) = x^3 - 3x$
   (iii) $f(x) = x^3 - 6x^2 + 9x + 15$ (iv) $g(x) = \frac{1}{x^2+2}$
   (v) $f(x) = x\sqrt{1-x},\ 0 < x < 1$
3. Prove that the following functions do not have maxima or minima.
   (i) $f(x)= e^x$ (ii) $g(x) = \log x$
   (iii) $h(x) = x^3 + x^2 + x + 1$
4. Find two positive numbers x and y shall that $x + y = 60$ and xy^3 is maximum.
5. Show that of all the rectangle inscribed in a given fixed circle, the square has the maximum area.
6. Show that the right circular cylinder of given surface and maximum volume is such that its height is equal to the diameter of the base.
7. Prove that the volume of the largest cone that can be inscribed in a sphere of radius R is $\frac{8}{27}$ of the volume of the sphere.

Integration as an Inverse process of Differentiation:

Integration is the inverse process of Differentiation. Instead of differentiating a function, we are given the derivation of a function and asked to find its primitive i.e., the original function. Such a process is called integration. Let us consider the following examples.

$$\frac{d}{dx}\left(\frac{x^3}{3}\right) = x^2$$

Here $\frac{x^3}{3}$ is the anti derivative or integral of x^2. Again we note that for any real number C, its derivative is zero and hence, we can write.

$$\frac{d}{dx}\left(\frac{x^3}{3} + C\right) = x^2$$

This integrals of the function are not unique C is treated to be an arbitrary constant. In general

$$\frac{d}{dx}[F(x)+C]=f(x), \qquad thus\ \{F+C,\ C\in R\}$$

Denotes a family of anti derivative of f

We introduce a new symbol, $\int f(x)dx$ which will represent the entire class of antiderivative or indefinite integral of f with repute to x.

We write $\int f(x)dx = F(x)+C$

Derivatives		**Integrals**
(i)	$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right)=x^n$	$\int x^n\,dx=\frac{x^{n+1}}{n+1}+C, n\neq -1$
(ii)	$\frac{d}{dx}(x)=1$	$\int dx = x+c$
(iii)	$\frac{d}{dx}(e^x)=e^x$	$\int e^x\,dx=e^x+c$
(iv)	$\frac{d}{dx}(log\|x\|)=\frac{1}{x}$	$\int \frac{1}{x}dx = log\|x\|+c$
(v)	$\frac{d}{dx}\left(\frac{a^x}{loga}\right)=a^x$	$\int a^x\,dx=\frac{a^x}{loga}+c$

Ex. Find the following integrals

(i) $\int \frac{x^3-1}{x^2}\,dx$ (ii) $\int \left(x^{2/3}+1\right)\,dx$

Solution. We have

(i) $\int \frac{x^{3-1}}{x^2}\,dx = \int x dn - \int x^{-2}\,dn$

$$=\frac{x^2}{2}+C_1+\frac{1}{x}-C_2$$

$$=\frac{x^2}{2}+\frac{1}{x}+C\ \ where\ C=C_1-C_2$$

(ii) $\int (x^{2/3}+1)\,dx = \int x^{2/3}dx+\int dx$

$$=\frac{3}{5}x^{5/3}+x+c$$

Exercise

1. Find integral $\int x^2\left(1-\frac{1}{x^2}\right)dx$
2. $\int\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2 dx$
3. $\int\frac{x^3+3x+4}{\sqrt{x}}\,dx$
4. $\int\sqrt{x}(3x^2+2x+3)\,dx$
5. $\int(1-x)\sqrt{x}\,dx$

Ex. If $\lim_{x\to 2}\left[\frac{x^n-2^n}{x-2}\right]=32$, then find the value of n

Solution :

$$\lim_{x\to 2}\left[\frac{x^n-2^n}{x-2}\right]=n.2^{n-1}$$

Or $\quad 32=n.2^{n-1}$

Or $\quad 4\times 8=n.2^{n-1}$

Or $\quad 4\times 2^3=n.2^{n-1}$

Or $\quad 4\times 2^{4-1}=n.2^{n-1}$

$\therefore \quad n=4$

Limits as $x\to\infty$:

We know, when $x\to\infty,\frac{1}{x}\to 0$ while evaluating limits infinity, put $x=\frac{1}{y}$

Ex: Evaluate $\lim_{x\to 2}\frac{2x+3}{x-5}$

Solution: Put $x=\frac{1}{y}$ if $x\to\infty, y\to 0$

$$\therefore\quad \lim_{x\to\infty}\frac{2x+3}{x-5}=\lim_{y\to 0}\frac{\frac{2}{y}+3}{\frac{1}{y}-5}$$

$$=\lim_{y\to 0}\frac{2+3y}{1-5y}$$

$$=\frac{2+0}{1-0}$$

$$=2$$

Continuity : We may say that a function is continuous at a fixed point if we can draw the graph of the function around that point without lifting the pen from the plane of the paper.

Definition : Suppose f is a real function on a sub-set of the real numbers and let 'C' be a point in the do main of f. Then f is continues at 'C' if.

$$\lim_{x \to C} f(x) = f(C)$$

More elaborately, if the left hand limit, right hand limit and the value of the function at $x = C$ exist and equal to each other, then f is said to be continuous of $x = C$, hence we may also rephrase the definition of continuity as follows a function is continuous at $x = C$ if the function is defined at $x = c$ and if the value of the function at $x = c$ equals the limit of the function at $x = c$.

If f is not continuous at C, we say f is discontinuous at 'C' and 'C' is called a point of discontinuity of f.

Ex: Check the continuity of the function f is given by $f\ x\ = 2x + 3\ at\ x = 1$

Solution: First note that the function is defined at the given point $x = 1$ and its value is 5. Then find the limit of the function at $= 1$ clearly $\lim_{x\to1} (x) = \lim_{x-1} 2x + 3 = 2\ 1 + 3 = 5$

$$\text{Thus} = \lim_{x\to1} f(x) = 5 = f(1)$$

Hence f is continues at $x = 1$

Ex 2: Examine whether the function f is given by $f(x) = x^2$ is continuous at $x = 0$.

Solution: First note that the function is defined at the given point $x = 0$ and its value is '0'

UNIT – 2 **Algebra**

Introduction of Matrices : Suppose we wish to express the information that Radha has 15 notebooks. We may express it is [15] with the understanding that the number inside [ ] is the number of notebooks that Radha has now if we have to express that Radha has 15 notebooks and 6 Pens. We may express it as [15 6] with the understanding that the first number inside [ ] is the number of notebooks while other one is the number of Pens possessed by Radha. Let is now suppose that we wish to express the information of possession of notebooks and Pens by Radha and her two friends Fauzia and Simran which is as follows :

Radha	has	15	notebooks	and	6	Pens
Fauzia	has	10	notebooks	and	2	Pens
Simran	has	13	notebooks	and	5	Pens

Now this could be arranged in the tabular forms as follows :

	Notebooks	Pens
Radha	15	6
Fauzia	10	2
Simran	13	5

And this can be expressed as

$$\begin{array}{cc} \begin{bmatrix} 15 & 6 \\ 10 & 2 \\ 13 & 5 \end{bmatrix} & \begin{array}{l} \leftarrow \text{First row} \\ \leftarrow \text{Second row} \\ \leftarrow \text{Third row} \end{array} \\ \begin{array}{cc} \uparrow & \uparrow \\ \text{First Column} & \text{Second Column} \end{array} & \end{array}$$

	Radha	Faxzia	Simran
Notebook	15	10	13
Pens	6	2	5

Which can be expressed as :

$$\begin{array}{cc} \begin{bmatrix} 15 & 10 & 13 \\ 6 & 2 & 5 \end{bmatrix} & \begin{array}{l} \leftarrow \text{First row} \\ \leftarrow \text{Second row} \end{array} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \text{First Column} & \text{Second Column} & \text{Third Column} \end{array} & \end{array}$$

An arrangement or display of the above kind is called Matrix.

Definition : A matrix is an ordered rectangular array of numbers or functions. The number or function are called the elements or the entries of the matrix.

$$A = \begin{bmatrix} -2 & 5 \\ 0 & \sqrt{5} \\ 3 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 2+1 & 3 & -\frac{1}{2} \\ 3-5 & -7 & 2 \\ \sqrt{3} & 5 & \frac{5}{7} \end{bmatrix}$$

$$C = \begin{bmatrix} 1+x & x^3 & 3 \\ cosx & sinx+2 & tanx \end{bmatrix}$$

The horizontal lines of elements are said to rows of the matrix and the vertical lines of elements are said to columns of the matrix.

A matrix having x rows and n columns is called a matrix of order $m \times n$. In general an $m \times n$ matrix has the following rectangular array.

$$\begin{pmatrix} a_{11} & a_{12} --- & a_{1n} \\ a_{21} & a_{22} --- & a_{2n} \\ a_{c1} & a_{c2} --- & a_{in} \\ a_{m1} & a_{m2} --- & a_{mn} \end{pmatrix} m \times n$$

Or $A = [aij]mxn, 1 \leq i \leq m, 1 \leq j \leq n, \quad i, f \in N.$

The number of elements in an $m \times n$ matrix will be equal to mn.

Square Matrix : A matrix in which the number of rows are equal to the number of columns is said to be a square matrix. This is an $m \times n$ matrix is said to be a Square matrix if $m = n$ and is known as a Square matrix of order n.

Equality of Matrix : Two matrices $A = [aij]$ and $B = [bij]$ are said to be equal if

(i) They are of the same order.
(ii) Each element of A is equal to the corresponding element of B that is $aij = bij$ for all i and j

Operations on Matrices :

1. **Addition of matrices:** The sum of two matrices is a matrix obtained by adding the corresponding elements of the given matrices. The two matrices have to be of the same order.

   In general it $A = [aij]$ and $B = [bij]$ are the matrices of the same order, say $m \times n$ Then the sum of the two matrices A and B is defined as a matrix $C = [cij]_{mxn}$ where $cij = aij + bij$ for all possible values of i and j.

2. **Multiplication of a matrix by a Scalar :** In general we may define multiplication of a matrix by a scalar as follows if $A = [aij]_{mxn}$ is a matrix and K is a scalar. Then KA is another matrix which is obtained by multiplying each element of A by the scalar K.

In other words $KA = K[aij]_{mxn}$
$= [kaij]_{mxn}$

That is $[i,j]^{th}$ element of KA is $K\ aij$ for all possible value of i and j.

Negative of a Matrix : The negative of a matrix is obtained by $-A$. We define $-A = (-1)\ A$.

Difference of Matrices : if $A = [aij], B = [bij]$ are two matrices of the same order say $m \times n$ Then difference $A - B$ is defined as a matrix $D = [dij]$ where $dij = aij - bij$ for all values of i and j in other words $D = A - B = A + (-1)B$.

Multiplication of Matrices : For multiplication of two matrix A and B the number of columns in A should be equal to the number of rows in B. The product of two matrices A and B is defined if the number of columns of A is equal to the number of rows of B. Let $A = [aij]$ be an $m \times n$ matrix and $B = [bij]$ be an $n \times p$ matrix. Then the proudest of the matrices A and B is the matrix C of order $m \times p$ to get the $(i,k)^{th}$ element C_{ik} of the matrix C, we take the i^{th} row of A and K^{th} Column of B, multiply them element wise and take the sum of all these product in other words if $A = [aij]_{mxn},\ B = [bjk]_{nxp}$ then the i^{th} take the sum of all these products in other words if $A = [aij]_{mxn}, B = [bjk]_{nxp}$ then i^{th}

row of A is $[ai_1\ ai_2 -\ a_{in}]$ and the K^{th} column of B is $\begin{bmatrix} b1k \\ b2x \\ ) \\ bnk \end{bmatrix}$ then

$cik = ai_1\ b_1k + ai_2\ b_2k + \cdots + ain\ bnk$

$= \sum_{j=1}^{n} aij\ bjk$

The matrix $C = [cik]_{mxp}$ is the product of A and B.

Ex. $C = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 7 \\ -1 & 1 \\ 5 & -4 \end{bmatrix}$

$CD = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 7 \\ -1 & 1 \\ 5 & -4 \end{bmatrix}$

$= \begin{pmatrix} (1)\ (2) + (-1)\ (-1) + (2)\ (5) & (1)\ (7) + (-1)\ (1) + 2\ (-4) \\ (0)\ (2) + (3)\ (-1) + (4)\ (5) & (0)\ (7) + (3)\ (1) + 4\ (-4) \end{pmatrix}$

$= \begin{bmatrix} 13 & -2 \\ 17 & -13 \end{bmatrix}$

Note : If AB defined then BA need not be defined. In particular if both A and B are square matrices of the same order, then both AB and BA are defined.

Determinant : To every square matrix $A = [aij]$ of order n, we can associate a number (real of complex) called determinant of the square matrix A, where $aij = (i, j)^{th}$ element of A

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant of A is written

As $|A| = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \det(A)$

Note : 1. Only Square matrices have determinants.

2. Transpose of a Matrix-Matrix obtained by interchanging the rows and columns of A is called the transpose of A.
3. Invertible Matrices - If A is a Square matrix of order m and if there exists another square matrix B of the same order such that $AB = BA = I$, then B is called the inverse matrix of A and is denoted by A^{-1}.
4. Adjoin of a matrix – The adjoin t of a square matrix $A = [aij]_{nxn}$ is defined as the $A = [aij]_{nxx}$ transpose of matrix where Aij is the cofactor of the element aij. Adjoin of the matrix A is defined by $adj\ A$.
5. A Square matrix A is said to be singular if $|A| = 0$.
6. A Square matrix A is said to be non-singular if $|A| \neq 0$.

Applications of Determinants and Matrices : Solution of system of liner equations using inverse of a matrix.

$$a_{1x} + b_{1y} + c_{1z} = d_1$$

$$a_{2x} + b_{2y} + c_{2z} = d_2$$

$$a_{3x} + b_{3y} + c_{3z} = d_3$$

Lel- $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $X = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

Then the system of equations can be written as AX=B

i.*e.*,

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Case 1 : If A is a non-singular matrix, then it's inverse exists Now

AX=B

$$A^{-1}(AX) = A^{-1}B$$

Or $$(A^{-1}A)X = A^{-1}B$$

Or $$1X = A^{-1}B$$

$$X = A^{-1}B$$

This matrix equations provides unique solution for the given system of equations as inverse of matrix is unique.

Case 1 : If A is a singular matrix, then $|A|0$ In this case, we calculate $(adj\ A)B$ if $(adj\ A)B \neq 0$ (0 being zero matrix) then the solution does not exist and the system of equations is called inconsistent. If $(adj\ A)B = 0$, then system may be either consistent or inconsistent according as the system have either infinitely many solutions or no solutions.

Ex.: Solve the system of equations

$$2x + 5y = 1$$

$$3x + 2y = 7$$

Solution: The system of equations can be written in the form $AX = B$ where

$$A = \begin{bmatrix} 2 & 5 \\ 3 & 2 \end{bmatrix}, \qquad X = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

Now $|A| = 11 \neq 0$, Here A is non-singular matrix and so has a unique solutions.

Note that $$A^{-1} = -\frac{1}{11}\begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix}$$

Note that $$X = A^{-1}B = -\frac{1}{11}\begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

i.e $$\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{11}\begin{bmatrix} -33 \\ 11 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

Hence $x = 3, y = -1$

Ex.: Solve the following system of equations by matrix method.

$$3x - 2y + 3z = 8$$
$$2x + y - z = 1$$
$$4x - 3y + 2z = 4$$

Solution : The system of equations can be written in the form $AX = B$ where

$$A = \begin{bmatrix} 3 & -2 & 5 \\ 2 & 1 & 2 \\ 4 & -4 & 2 \end{bmatrix}, \qquad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$$

We see that

$$|A| = 3(2-3) + 2(4+4) + 3(-6-4) = -17 \neq 0.$$

Hence, A is non-singular and so its inverse exists. Now

$$\begin{array}{lll} A_{11} = -1 & A_{12} = -8 & A_{13} = -10 \\ A_{21} = -5 & A_{22} = -6 & A_{23} = 1 \\ A_{31} = -1 & A_{32} = 9 & A_{33} = 7 \end{array}$$

Therefore $\quad A^{-1} = -\frac{1}{17}\begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 2 \\ -10 & 1 & 7 \end{bmatrix}$

So $\quad A^{-1} = -\frac{1}{17}\begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 2 \\ -10 & 1 & 7 \end{bmatrix}\begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$

$i.e \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{17}\begin{bmatrix} -17 \\ -34 \\ -51 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Hence $\quad x = 1, y = 2,\ z = 3$

<u>Exercise</u>

Solve system of liner equations, using matrix method.

(i) $5x + 2y = 4$

$7x + 3y = 5$

(ii) $x - y + z = 4$

$2x + y - 3z = 0$

$x + y + z = 2$

(iii) $2x + 3y + 3z = 5$

$x + 2y + z = -4$

$3x - y - 2z = 3$

(iv) $x - y + 2z = 7$

$3x + 4y - 5z = -5$

$2x - y + 3z = 12$

UNIT -3

LOGICAL REASONING

Binary Numbers:

The base of binary number system is 2 because it consists of two different digits or symbols i.e. 0 and 1. The digits 0 and 1 are called binary digits commonly abbreviated as bits. Each position in a binary number represents a power of the base (2) In this system, the rightmost position is the units (2^0) position, the second position from the right is the (2^1) position and the (2^2) position (2^3) position, (2^4) position and so on.

Thus, the decimal equivalent of the binary 11010 is :

$(1x2^4) + (1x2^3) + (0 \times 2^2) + (1x2^1) + (0x 2^0)$

or 16 + 8 + 0 + 2 + 0

or 26

Thus we can write :

$(11010)_2 = (26)_{10}$

Binary Arithmetic :

(i) Binary Addition : As binary number system consists of only 2 digits i.e. 0 and 1, there are only four possible combination the binary addition, they are :

$0 + 0 = 0$

$0 + 1 = 1$

$1 + 0 = 0$

$1+1 = 0$ and 1 carry over to next higher column.

Ex.1

```
              1 0 1
          +   1 0 1
Carry   ←   1 0 1 0
```

Ex. 2

```
              1 0 1 0 0 1
          +   1 1 0 1 1 0
Carry   ←   1 0 1 1 1 1 1
```

Binary Subtraction :

There are four possible combnations for binary subtraction, they are :

0 - 0 = 0

1 - 0 = 1

1 - 1 = 0

0 - 1 = 1 with a borrow from the next column.

Ex.

```
   1 1 1 0
 - 1 0 0 0
 ---------
   0 1 1 0
```

Binary Multiplication :

Binary multiplication is similar to normal multiplication

0 x 0 = 0

0 x 1 = 0

1 x 0 = 0

1 x 1 = 1

```
Ex.        1 1 0 1
             x 1 1
        ------------
           1 1 0 1
         1 1 0 1
        ------------
         1 0 0 1 1 1
        ------------
```

Binary Division :

The division by zero is meaningless in binary division similar to other number system. There are only two possible combinations for binary division, they are :

0/1 = 0

1/1 = 1

```
            11
         ---------
    111 / 101111
            1 1 1
         ---------
          1 0 0 1
            1 1  1
         ---------
            1  0 1  remainder
```

Logical Operators :

Logical operators are the connections used to form compounded statements since these operaters are operated on logical values 0 and 1, that is why these operators are called logical operators. There are three logical operators :

1. AND operater 2. OR operator 3. Not operator

Logical Operator	Symbols
AND	$\wedge$
OR	V OR +
NOT	- or '

AND OPERATOR :

And operator is used to perform logical multiplication between the true value of logical statements to give the final value as output AND operator is represented by the symbol '$\wedge$'. And operator gives the logical value as "true or 1". Only when the logical values for all the input logical statements are true. For example if there are two logical input statements A and B then the final logical value will be true only when A and B are true. For N Number of inputs the possible combinations are 2^N. For two input logical statements A and B, 2^2 possible combinations are as follows:

A	B	A B
0	0	0
0	1	0
1	0	0
1	1	1

Truth Table

OR Operator :

OR operator is used to perform logical addition between the true values of logical statements to give the final value as output OR operator is represented by the symbol "V or +", OR operator gives the logical values as "true " or 1 when the logical value for any one logical input statements is 1. It means the only case when output will be 0 is when all inputs are 0 e.g. if A and B are the two logical input statements, the final value will be 1 when either of the statements of both the statements are true.

A	B	A + B
0	0	0
0	1	1
1	0	1
1	1	1

Truth Table

NOT Operator :

NOT operator is used to perform logical negation. It is a Unary opertor because logical negation. It is a Unary operator because it operates on a single variable. NOT operators is also known as complementation operator or Inverse operator. NOT operator returns the opposite value as final output. It means if the input value is 1 then the output value will be 0 and if the input value is 0 then the output value will be 1.

NOT operator is represented by the symbols '-' or '.

X	X^1
0	1
1	0

Truth Tables :

The tabular representatives of truth values of a compound statements based on the truth values of the prime consecutiveness of statement is called 'Truth Table'.

Truth table consists of horizontal line (rows) and vertical lines (columns). If a compound statement consists of N statements, the number of rows will be 2^N. The number of columns in a truth table depends upon the n umber of relationship between these statement.

We observe the following facts :

(i) If p and q both have truth value T, then $p \wedge q$ had truth value T.

(ii) If p has truth value T and q has truth value F, then $P \wedge q$ has truth value F.

(iii) If p has truth value F and q has truth value T, then $p \wedge q$ has truth value F.

(iv) If both p and q have truth value F, then $p \wedge q$ has truth value F.

If follows from the above discussion that $p \wedge q$ is true only when each one of p and q is true otherwise $p \wedge q$ is false.

Truth Table for Conjunction

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

In the case of conjunction, we can easily see that $p \vee q$ is true only when at least one of p and q is true, otherwise it is false.

Truth Table for Disjunction

p	q	p $\vee$ q
T	T	T
T	F	T
F	T	T
F	F	F

If p is true, then p is false; and if p is false, then p is true

Truth Table for Negation

p	~ p
T	F
F	T

Ex.1 Give the truth table for the statement $\sim (p \vee q)$

Solution

p	q	$\sim p$	$\sim (p \vee q)$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

Ex.2 Give the truth table of $\sim (p \wedge q)$

Solution

p	q	$\sim p$	$\sim (p \wedge q)$
T	T	F	F
T	F	F	F
F	T	T	T
F	F	T	F

Number Series :

In the number series the terms follow a certain pattern throughout the series. We have to study the given series, identity the pattern followed in the series and either complete the given series with the most suitable alternative or find the wrong terms in the series.

Find the missing term in each of the following series :

Ex.1. 1, 6, 15, ?, 45, 66, 91

(a) 25 (b) 26 (c) 27 (d) 28

Ex.2 2 5, 9, 19, 37, ?

(a) 73 (b) 75 (c) 76 (d) 78

Ex. 3 4, 8, 28, 80, 244, ?

(a) 278 (b) 428 (c) 628 (d) 728

Ex. 4 1, 4, 27, 16, ?, 36, 343

(a) 25 (b) 87 (c) 120 (d) 125

Triangular Pattern Series :

Sometimes the differences between the consecutive terms of a series, again form a series. The different between the consecutive terms of the new series so formed, again form a series. This pattern continues till we attain a uniform differences between the consecutive of the series.

Ex. 5 Find the missing term in the series,

3, 20, 63, 144, 275, ?

(a) 354 (b) 468 (c) 548 (d) 554

Solution :

Series I :	3,	20,	63,	144,	275,	?
Series II:	17	43	81	131	?	
Series III:	26	38	50	?		
Series IV:	12	12				

Clearly the pattern in series III is +12. So missing term in series III = 50 + 12 = 62.

Missing term in series II = 131 + 62 = 193

Missing term in series I = 275 + 193 = 468

Elementary Idea of Progression :

1. Arithmetic Progression (A.P.) - The progression of the form a, a+d, a+2d, ... is known as A.P. with first term = a and common different = d. In an A.P. we have nth term = a + (n-1).d

2. Geometric Progression (:G.P.) - The progression of the form a, ar, ar^2, ... is known as a G.P. with the first term = a and common ratio = r.

In a G.P., we have the nth term = ar^{n-1}.

Ex. 6 In the series 357, 363, 369, ... what will be the 10^{th} term?

(a) 405 (b) 411 (c) 413 (d) 417

The given series in A.P. in which a = 357 and d = 6.

Therefore, 10th term = a + (10-1).d = a + 9d

= (357 + 9 x 6) = (357 + 54) = 411

Ex. 7 In the series 7, 14, 28, ... what will be the 10th term?

(a) 1792 (b) 2456 (c) 3584 (d) 4096

given series is a G.P. in which a = 7 and r = 2

therefore 10th term = $ar^{(10-1)} = ar^9 = 7 \times 2^9 = 7 \times 512 = 3584$

Coding-Decoding :

A code is a 'system of signals'. Therefore, coding is a method of transmitting a message between the sender and the receiver without a third person knowing it.

Letter Coding :

In the these questions, the letter in a word are replaced by certain other letters according to a specific rule to form its code. The candidate is required to detect the coding pattern rule and answer the questions accordingly.

To form the code for another word (Coding)

Exp. 1 In a certain code, TEACHER is written as VGCEJGT. How CHILDREN written in that code?

(a) EJKNEGTP (b) EJKNFITP

(c) EJKNFGTO (d) EJKNFTGP

Solution : Clearly, letter in the word TEACHER is moved from steps forward to obtain the corresponding letter of the code

T	E	A	C	H	E	R
+2↓	+2↓	+2↓	+2↓	+2↓	+2↓	+2↓
V	G	C	E	J	G	T

Similarly we have

C	H	I	L	D	R	E	N
+2↓	+2↓	+2↓	+2↓	+2↓	+2↓	+2↓	+2↓
V	J	K	N	F	T	G	P

so the desired code is EJKNFTGP

Number/Symbol coding :

In these questions either numerical code values are assigned to a word or alphabetical code letters are assigned to the numbers. The candidate is required to analyse the code as per the questions.

Exp. If MACHINE is coded as 19-7-9-14-15-20 -11), how will you code DANGER?

(a) 11-7-20-16-11-24

(b) 13-7-20-9-11-25

(c) 10-7-20-13-11-24

(d) 13-7-20-10-11-25

Solution : Clearly, every letter is assigned a numerical code obtained by adding 6 to the numerical denoting the position of that letter in the English alphabet.

Thus A is coded as (1+6) i.e 7, B as (2+6) i.e. 8, C as (3+6) i.e. 9, (13+6) i.e. 19, ... z as (26+6) i.e. 32.

Since D, A. N, G, E, R are 4th, 1st, 14th, 7th, 5th and 18th letters in the English alphabet, so their respective course are (4+6), (1+6), (14+6), (7+6), (5+6), 18+6) i.e. 10, 7, 20, 13, 11, 24 so the code for DANGER is 10-7-20-13-11-24.

Odd Man Out :

To assort the items of a given group on the basis of a certain common quality they posses and the spot the stranger or 'odd man out'. In this test, you are given a group of certain items out of which all except one are similar to one another in some manner. The candidate is required to choose this one item which does not fit into the given group.

Choose the word which is least like the other words in the group.

Exp.1 (a) Zebra (b) Lion (c) Tiger (d) Horse (e) Giraffe

Solution : Here, all except Horse, are wild animals, while Horse can be domesticated.

Hence the answer is (d)

Exp. 2 (a) Copper (b) Zinc (c) Bross (d) Aluminium (e) Iron

Solution : Here, all except Bross are metals, while Bross is an alloy. Hence the answer is (c).

Ex. 3 (a) January (b) May (c) July (d) August (e) November

Solution: Here, all except November are months having 31 days, while November has 30 days. Hence the answer is (c).

Exp. 4 (a) Pistol (b) Sword (c) Gun (d) Rifle (e) Cannon

Solution : Here, all except sword are fire arms, and can be used from a distance. Hence the answer is (b).

Exp. 5 Choose the number of pair group which is different from others:

(a) 50-66 (b) 32-48 (c) 64-80 (d) 63-77

Sol. Clearly, in each of the p airs except (d), the second number is 16 more than the first. Hence the answer is (d).

Exp. 6 (a) 42 : 4 (b) 36 : 6 (c) 32 : 2 (d) 15 : 5

Sol.: In each of the p airs except (a). The first number is a multiple of the second. Hence the answer is (a).

Exp.7 Choose the group of letters which is different from others.

(a) BD (b) IK (c) PN (d) SU (e) WY

Clearly the answer is (c)

Direction Test :

A successive follow up of directions is formulated and candidate is required to ascertain the final direction or the distance between two points. The test is meant to judge the candidates ability to trace and follow correctly and since the direction correctly.

Exp. : A man is facing west. He turns 45^0 in the clockwise direction and then another 180^0 in the same direction and then 270^0 in the anti-clockwise direction. Which direction is he facing now?

(a) South (b) North-west (c) West (d) South-west

Answer is (d)

Exp. 2 : If you are facing north-east and move 10m forward, turn left and move 7.5m, then you are :

(a) north of your initial position.

(b) south of your initial position

(c) east of your initial position

(d) 12m from your initial position

(e) Both (c) and (d)

Exp. 3 : A man is facing south. He turn 135^0 in the anticlockwise direction and then 180^0 in the clockwise direction. which direction is he facing now?

(a) North-east (b) North-west (c) South-east (d) South-west

Exp. 4: A man is facing north-west. He turns 90^0 in the clockwise direction and then 135^0 in the anticlockwise direction. which direction is he facing now?

(a) East (b) West (c) North (d) South

Blood Relations :

In this type of questions a round about description is given in the form of certain small relationships and you are required to analyse the whole chain of relations and decipher the direct relationship between the persons concerned.

Exp. 1 : Pointing towards some person, a men said to a woman, "His mother is the only daughter of your father". How is the women related to that person?

(a) daughter (b) sister (c) mother (d) wife

Sol.: The only daughter of woman's father is she herself. So, the person is women's son i.e. the woman is the person's mother. Hence the answer is (c).

Ex.2 : Anil introduces Rohit as the son of the only brother of his father wife. How is Rohit related to Anil?

(a) Cousin (b) Son (c) Uncle (d) son-in-law (d) brother

Sol.: The relation may be analyzed as follows:

Father's wife- Mother, Mother's brother- Uncle, Uncle's son- Cousin.

so, Rohit is Anil's Cousin, Hence the answer is A.

Exp 3 : Read the following information carefully and answer the question given below :

'A+B' means 'A is the son of B', 'A-B' means 'A' is the wife of B', 'A X B' means 'A is the brother of B, 'A ÷ B' means 'A is the mother of B' and 'A = B' means 'A is the sister of B'.

What does P+R - Q mean?

(a) Q is the father of P (b) Q is the son of P.

(c) Q is the uncle of P (d) Q is the brother of P.

P+R-Q, means P is the son of R who is the wife of Q i.e., Q is the father of P.

Hence the answer is (a).

SYLLOGISM :

For given questions, a statement is followed by two conclusions. Point out the correct conclusion :

(a) If only conclusion (I) is true.

(b) If only conclusion (II) is true.

(c) If both the conclusion are true.

(d) If neither of the conclusion is true

Ex. 1: Statement : The increase in adult literacy will lead to country's development and progress.

Conclusion :

I. The educated people offer less resistance to new innovations.

II. The literacy rate is higher in developed countries.

Sol.: Answer is (c) : The literate people will understand new innovations easily. Also developed countries have higher literacy rate so both the conclusions are true.

Exp. 2 : Statement : Education is in the concurrent list. The state Government cannot bring reforms in education without the consent of the central government.

Conclusion :

I. For bringing about quick reforms in education, it should be in the state list.

II. States are note willing to bring about quick reforms in education.

Sol. : Neither of the given conclusion follows:

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UNIT - 4

SHARE, DEBENTURES, STOCKS

Joint Stock Company :

To start an industry or a big business a large amount of money is required if an individual does not have sufficient money, then some persons associate together and prepare a detailed plan of the project with the help of some experts in that particular field. They also frame some rules and regulation regarding its functioning. These are then registered under the Indian companies Act. The company so formed is called a joint stock company the total amount of money required for the project is called the capital.

Shares :

A joint stock company divides the required capital into equal small units. Each unit is called share. The company then issues a prospectus, explaining the plan of the project and invites the general public to invest money in the proposal project by purchasing the shares of the company. Those persons who accept the terms and condition of the company and consider the investment profitable, apply for these shares. The company resure the right whether to allot shares to a person or not This situation generally arises when the n umber of applicants is very large in comparison to number of shares. When one is allotted shares by the company and has paid the money prescribed by the company for the shares, the company issues certification indicating the number of shares allotted to the person and the value of each share. These certificates are called share certificates. The person who subscribes in shares are called share holder.

Face Value :

The value for which a share is issued by a company is called the face value of the share.

The face value of a share is printed on the Share Certificate and is also known as the nominal value or par value of the share.

Market Value :

Like any other commodities shares can be sold and purchased in a market, called stock exchange when we sell a share in the market it may fetch a value more than the face value or less than the face value depending upon market. conditions and so many other factors affecting the market. The value for which a share is available in the market is called the market value of the share.

Ex. 1 If Rs. 100 share is quoted at 75 premium, then its market value is Rs. (100 + 75) = Rs. 175

Ex. 2 If Rs. 100 share is quoted at 10 discount, then the market value of the share is Rs. (100-10) - Rs. 90.

Brokerage :

Shares are sold and purchased in a market, called stock exchange. The sale and purchase of shares is, generally done through agents called share- brokers or simply brokers. Brokers charge commission from buyers as well as sellers. The broker's commission is called brokerage Brokerage is calculated on the market value of the share.

Types of Shares :

There are two types of shares :

(i) Preferred shares (ii) Common shares or ordinary Shares.

Preferred Shares :

In preferred shares, the share holders receive a specified percent of the profits as dividend. Note that preferred share holders are paid dividend only if the company has profits after paying working expenses and taxes.

Common or ordinary shares :

Common or Ordinary shares are those shares which are paid dividend only when profits are left after preferred share holders have been paid dividend at specified rate. Some time common share holders do not get any dividend because no profit is left for them after paying for working expenses, taxes and preferred share holders.

Ex. A company issued shares at 10% premium satish applied for 1000 shares but was allotted 500 shares of this company. Find his investment if the face value of a share is Rs. 100.

Solution : Face value of a share = Rs. 100

Premium Percentage = 10%

Premium on one share = Rs. 10

Market value of one share = Rs. (100 + 10) = Rs. 110

Price of 500 share = Rs. (500 x 110) = Rs. 55000

Ex. 2. Find the investment in buying 450 shares of Rs. 100 each at 5 discount.

Solution. Face value of a share = Rs. 100

Market value of a share = Face value - discount

= Rs. (100-5) = Rs. 95

So, investment in buying 450 shares = Rs. (450 x 95)

= Rs. 42750.

Debentures :

Some times a company needs money for further expanction and diversification of its activities. For this, the company may not issue new shares but may borrow this money from the public/ share holders for a fixed period of time at a fixed rate of interest. In such a case, the company issues debentures. The whole amount of debt needed by the company is divided into equal small units These units are called debentures. Like share certificates a company issues debentures certificates. Debentures holders of a company get a fixed rate of interest generally payable half yearly or yearly, irrespective of the profit or loss incurred by the company After expiry of the fixed period, the company repays the debenture money to its holders.

Like shares, debentures are also sold and purchased in the market. However, the interest on a debenture is always calculated on its face value and the brokerage is calculated on its market values.

Ex. : Compute the annual yield percent on 12% debentures of face value of Rs. 100 each and available at Rs. 80 each.

Solution : It is given that :

Face value of a debenture = Rs. 100

Market value of a debenture = Rs. 80

Rate of interest on debentures = 12%

Now, by investing Rs. 80, annual interest = Rs. 12

By investing Rs. 100, annual interest = Rs. $\frac{12}{80} x\ 100 = Rs.15$

Hence annual yield = 15%

Ex.2: which is better investment : 15% debentures at 8% premiu m or 14% debentures at 4% discount?

Solution: Let the face value of each debenture be Rs. 100.

Now, 15% debentures at 8% premium means:

Face value of a debenture = Rs. 100

Market value fo a debentures = Rs. 108

Rate of interest = 15%

Interest on debenture of Rs. 100 = Rs. 15

Thus, by investing Rs. 108 annual interest = Rs. 15

By investing Rs. 100, annual interest = Rs. $\left(\frac{15}{108} x\, 100\right)$

= Rs. 13.88

14% debentures at 4% discount means :

Face value of a debenture = Rs. 100

Market value of a debenture Rs. (100-4) = Rs. 96

Rate of interest = 14%

Interest on a debenture of Rs. 100 = Rs. 14

Thus, by investing Rs. 96, annual interest = Rs. 14

By investing Rs. 100, annual interest = Rs. $\left(\frac{14}{96} x\, 100\right)$

= Rs. 14.58

Clearly second investment yields more interest.

Hence it is a better investment.

Stocks :

Sometimes- Government requires money to meet the expenses of a certain project or a big work of public utility. In such cases, Government raises a loan from the public at the fixed rate of interest and issues Bonds or Promissory Notes as an acknowledgement of the debt. These bonds are generally for Rs. 100 and in some cases for Rs. 500 and Rs. 1000. The government pays interest at a fixed rate to the holder of these bonds. The interest is paid on the value printed on the bond certificate. If a person buys a bond of Rs. 100 on which 10% interest has been fixed by the government then the holder of such a bond is said to have a Rs. 100 stock at 10%

Usually, the government repays the loan at a date fixed at the ti me of issue of bonds. This date is called the maturity date. In case h older of a stock needs money before the date of maturity he cannot ask the government to pay back his money. Bus he can sell his stock to some other person (s), where by his claim to interest is transferred to that person(s). The sale and purchase of stocks is done exactly in the same way as is done for shares.

Ex. Find the cost of Rs. 5000 of 7% stock at 92.

Solution : The expression 7% stock at 92 means that Rs. 100 stock can be purchased for Rs. 92 and it pays a dividend of Rs. 7 per annum.

Cost of Rs. 100 stock = Rs. 92

= Cost of Re. 1 stock = Re. $\frac{92}{100}$

= Cost of Rs. 5000 stock = Rs. $\left(\frac{92}{100} x\ 5000\right)$

= Rs. 4600

Exercise

1. Find the investment in buying 525 shares of Rs. 100 each at 12 premium.
2. Find the percent income of buyer on 6% debentures of face value Rs. 100 available in the market for Rs. 150.
3. Find the annual yield percent on 16% debentures of face value Rs. 100 each and available at Rs. 80 each.
4. Find the cost of Rs. 7000 of 15% stock at 105.
5. How much 11% stock at 97 can be bought by investing Rs. 24250?

EQUATED MONTHLY INSTALMENT (EMI) :

EMI is the fixed amount payable monthly throughout the repayment period of a loan by the borrower to the lending institution. It consists of a portion of interest as well as a principal. EMI system of loan repayment has the following features.

1. Each installment contains both components of principal repayment and interest charges.

2. Interest in calculated on reducing balance method

3. Interest component is higher in the beginning and progressively lower towards the end. That mean, the principal component of an EMI is lower during initial periods and higher during later periods.

4. The amount of EMI depends on :

\(i) The period of compounding i.e. whether the compounding is yearly, half yearly, quarterly or monthly. If the compounding is more frequent, then the amount of EMI would be higher and vice- versa.

(ii) The rate of interest

(iii) Period of repayment if the repayment period is more, then EMI would be lower and Vice versa

$$EMI = \frac{p \ x \ r \ \ x \ (1+r)^n}{[(1+r)^n \ -1)}$$

where p = principal i.e. loan amount

r = rate of interest per month

n = no. of installments

Ex. Loan amount = Rs. 10,00,000/-

Interest rate = (% rate)/ 12 months

= 11%/ 12 months

= 0.0091

Loan period (N) = 15 years = 180 months

$$EMI = (10{,}00{,}000 \ x \ 0.0091) \ \ x \ \ \frac{(1+0.0091)^{180}}{(1+0.0091)^{180} - 1}$$

= Rs. 11,365.96 which has to be paid every month towards principal and interest amount

Calculations of Returns :

All financial decisions involve same risk one may expect to get a return of 15% per annum in his investment but the risk of not able to achieve 15% return will al ways be there. Return is simply a reward for investing as all investing involves some risk.

A debt investment is a loan, and the return is just the loans interest rate. This is simply the ratio of the interest paid to the loan principal

$$K = \frac{\text{int}\, erest\ paid}{loan\, amount}$$

This formulation leads to the convenient idea that a return is what the investor receives divided by what he or she invests. A stock investment involves the receipt of dividents and a capital gain (loss). If a stock investment is held for one year. The return can be written as

$$K = \frac{D_1 + (P_1 - P_0)}{Po}$$

Here P_0 is the price today, while P_1 and D_1 are respectively the price and divided at the end of the year.

Return on Investment (ROI) :

R.O.I. is a performance measure used to evaluate the efficiency of an investment or compose the efficiency of a number of different investments. ROI tries to directly measure the amount of return on a particulars investment, relative to the investment's cost. To calculate ROI, the benefit (or return) of an investment is divided by the cost of the investment. The result is expressed as a percentage or a ratio.

$$ROI = \frac{(Current\ Value\ of\ Investment - \cos t\ of\ Investment)}{Cost\ of\ Investment}$$

Formula for rate of return

$$Rate\ of\ return = \frac{Endings\ value\ of\ investment - beginning\ value\ of\ Investment}{Bigning\ value\ of\ investment}$$

Ex. Suresh is a retail investor and decides to purchase 10 shares of company A at a per unit price of Rs. 20. Suresh holds onto shares of company A for 2 years. In that time frame, company A paid yearly dividend of Rs. 1 per share. After holding them for 2 years. Suresh decides to sell all shares of company A at an ex- dividend price of Rs. 25. Suresh would like to determine the rate of return during the two y ears he owned the shares.

To determine the rate of return, first calculate the amount of dividends he received over t he two year period.

10 shares X (Rs. 1 annual dividend X2)

= Rs. 20 in dividends from 10 shares

Next, calculate how much he sold the shares for

10 shares X Rs. 25 = Rs. 250 (gain from selling 10 Shares)

Compound Annual Growth Rate :

Compounded Annual growth rate (CAGR) is a business and investing specific term for the smoothed annualized gain of an investment over a given time period. CAGR is not an accounting term, but remains widely used, particularly in growth industries or to compare the growth rates of two investments. CAGR is often used to describe the growth over a period of time of some element of the business for example revenue, units delivered, registered users etc.

Formula CAGR (t_0, t_n) = $\left(\frac{v(tn)}{v(to)}\right)^{\frac{1}{tn-to}}$

v(to) : start value, v(tn) : finish value

tn - to : number of years

Ex. Suppose the revenue of a company for four years, v(t) in above form ula, have been :

year	2004	2005	2006	2007
Revenues	100	115	150	200

E_n - to = 2007 - 2004 = 3

Then the CAGR of revenues over the three year period from the end of 2004 to the end of 2007 is :

$$CAGR\,(0,\,3) \;=\; \left(\frac{200}{100}\right)^{1/3} - 1$$

$$= 0.2599$$

$$= 25.99\%$$

Verification : If you multiply the initial values by (1+CAGR) three times (because we calculated for 3 years) you will get exactly the final value again. This is

$v(tn) = v(to) \times (1+ CAGR)^n$

for n = 3

= v(to) x (1+CAGR)x(1+ CAGR) x (1+CAGR)

= 100 x 1.2599 x 1.2599 x 1.2599

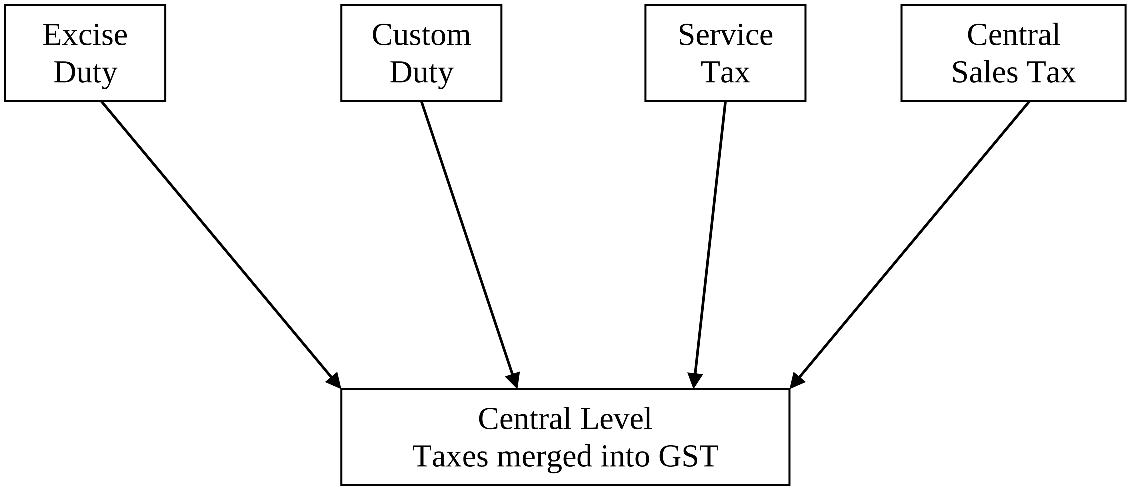
= 200

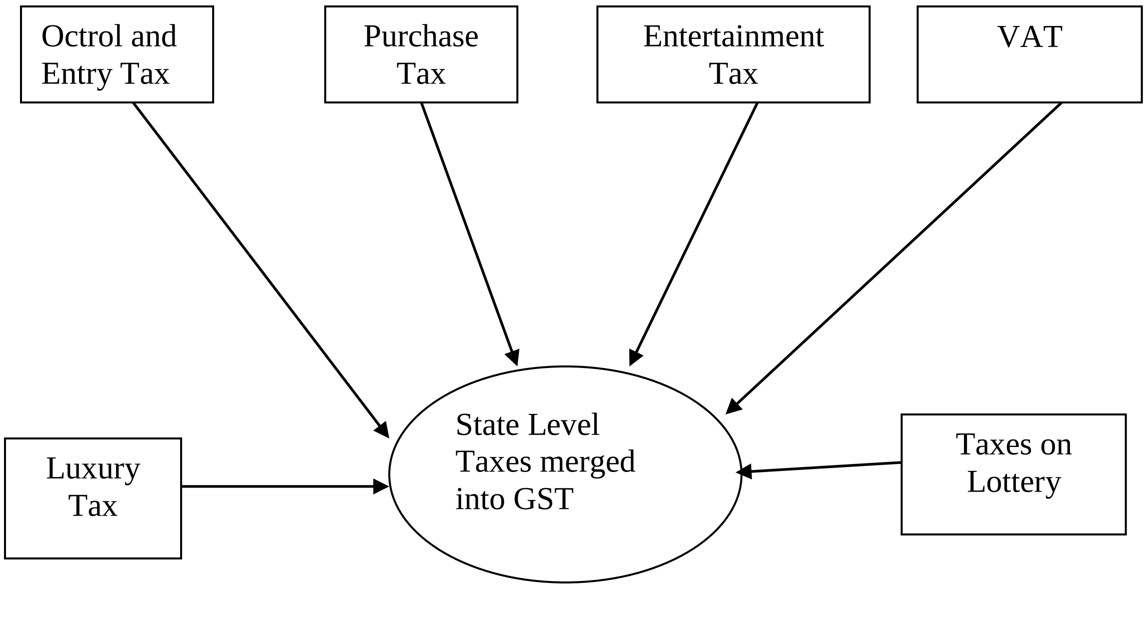
Meaning of GST :

Goods and service Tax (GST) is an indirect tax levied on the supply of goods and services. It is a nation wide tax seeking to unify several indirect taxes and is based on the principle of 'one Nation one tax'.

GST Act was passed in the parliament on 24th March 2017 and it came into effect from 1st July 2017

Taxes Merged into GAST-





GST is paid on purchase of goods and services and it is collected from customers on sale of goods and Services. GST paid (termed as input GST) is set off against GST collected (termed as output GST)

As such, GST paid on purchase (input GST) is not a cost for the purchaser but is an Asset since it can be set off against GST collected on sale (i.e. output GST). Similarly, GST collected on sales (output GST) is not an income of the seller but is a liability.

Types of Taxes under GST : There are 3 taxes applicable under GST :

(i) Central

GST (CGST)

(ii) State GST (SGST) or union territory GST (UTGST)

Both of these taxes are levied on intra- state sales i.e. within the same stat e. For example a dealer of Rajasthan sells goods to a dealer (or consumer) in Rajasthan worth Rs. 50,000. Suppose, the GST rate on the goods is 12%. This rate will comprise of CGST at 6% and SGST at 6%. The seller has to collect 12% of Rs. 50,000 i.e. Rs.6,000 out of which Rs. 3000 will be CGST which will be to the central Government and Rs. 3000 will in SGST which will go to the Rajasthan Government.

(ii) Integrated GST (IGST) : It is leived on inter- state sales i.e. sales of goods and service outside the state It is also levied an import of goods and services into India and exports of goods and services from India. For example, a dealer of Rajasthan Sells goods to a dealer in Madhya Pradesh worth Rs. 50,000 suppose the IGST rate is 12%. In such a case, the seller will charge Rs. 6000 as IGST and this entire amount will go to central government.

Concept of banking :

Among the various types of services offered by banks, taking deposits and providing loans are the basic ones. Apart from these, banks, render the following types ancillary services.

(i) Remittance of Funds- Banks help in transferring money from one place to another in a safe manner. They do this by issuing ddemand drafts, money transfer orders, and telegraphic transfers. Banks also issue traveller's cheques in home currency and also in foreign currency with the advent of technology, money transfer has become easy through internet and phone banking.

(ii) Safe deposit Lockers - Bank provide safety Lockers to customers to safely preserve their valuables.

(iii) Public Utility services- Through bank accounts customers can pay their telephone bills, electricity bills insurance premium and several other services.

DEPOSIT ACCOUNTS :

(i) Saving Bank Account - An Indian individual, either resident or non- resident, can open a Saving bank account with a minimum balance may vary from bank to bank. Such an account can be opened in joint name also. A saving bank account carries certain amount of interest compounded half- yearly. The rate of interest varies from bank to bank and time to time cheque books are issued to an account holder on demand.

Depositing Money in the Bank account- Money can be deposited in a bank account either by cash or through a duly filled p ay-in slip or challan

Demand Draft - Money can be deposited through demanded draft (i.e., bank draft). A bank draft is an order issued by a bank to its specified branch or to another bank to make payment of the amount to the party, in whose name the draft is issued.

Withdrawal of Money from Saving Bank Account

Money deposited in these accounts can be withdrawn by using withdrawal slips or cheques. Cheque books are issued only to those account holders who fulfill certain special requirements. Such as maintenance of minim um balance updated information related to the account.

Types of Cheques

(i) Bearer Cheque - A bearer cheque can be encashed by any one who possesses the cheque, though the person's name is not written on the cheque. There is a risk of wrong a person getting the payments.

If the word 'bearer' is crossed - out in a cheque, then the person whose name appears on it can alone encash. This type of cheque is known as 'order cheque'.

Crossed Cheque :

When two parallel lines are drawn at the left hand top corner of a cheque, it is called crossed cheque. The words A/c Payee may or may not be written between the two parallel lines. The payee has to deposit crossed cheque in his/ her account. The collecting bank collects the money from the drawer's bank and it is credited to the payee's account.

Calculation of Interest on Saving Account in Banks ;

The monthly minimum balances from January to the end of June are added. This total amount is called the (product) in bank. Interest i s calculated on this product and added to the opening balance on July 1st. In the same manner, the interest for the next half year is calculated and added to the opening balance on January 1st.

Interest is calculated by maintaining the following steps:

1. The least of the balances from the 10th day of a month to the last day of the month is considered as the balance for that month.

2. The sum of all these monthly balance is considered as the principle for calculating interest.

3. Interest = $\frac{\text{Principle } x \text{ rate of interest}}{12\ x\ 100}$

Current Account :

Current Account is convenient for business people, companies, government offices and various other institutions requiring frequent and large amount of monetary transactions. Bank do not given any interest on this accounts. There is no restriction on amounts deposited or withdrawn (i.e. on the number of transaction) as savings bank accounts.

Terms Deposit Account s: Term deposit accounts are of two types:

1. Fixed deposit accounts 2. Recurring deposit accounts

1. Fixed Deposit Accounts :

Customers can avail the facility of depositing a fixed amount of money for a difinite period of time. If money is withdrawn from these accounts before the specified time period, banks pay lesser interest than what was agreed upon the rate of interest payable varies with the period for which money is deposited in these accounts.

Recurring Deposit Accounts :

These accounts facilititate depositing a fixed amount per month for a time span of 6 months to 3 years and above. The time period is called maturity period. Recurring deposit accounts are helpful to people with low earning. They can save large amounts through regular and fixed savings.

Loans: Bank loans can be classified into the following three categories.

1. Demand Loan

2. Term Loan

3. Overdraft (ODs)

1. Demand Loan :

The borrower has to repay the loans on demand. The repayment of the loan has to be done within 36 months from the date of disbursement of the loan. The borrower has to execute a demand promissory note in favour of the bank, promising that he would repay the loan unconditionally as per the stipulation of the bank.

Terms Loan :

The borrower enters into an agreement with the bank regarding the period of loan and mode of repayment, number of installments etc. These loan are availed by purchasers of machinery, build houses etc.

Overdrafts (ODS) :

A current account holder enters into an agreement with the bank which permits him to draw more than the amount available in his account but upto a maximum limit fixed by the bank. These loans are availed by traders.

UNIT -5

PROBABILITY

Introduction

Probability is a measure of uncertainty of various phenomenon. We have obtained the probability of getting an even number in throwing a die as $\frac{3}{6}$ i.e. $\frac{1}{2}$. Here the total possible outcomes are 1, 2, 3, 4, 5 and 6. The outcome in favour of the event of getting are even number are 2, 4, 6. In general to obtain the probability of an event, the find the ratio of the number of outcomes favourable to the event, to we total number of equally likely outcomes. This theory of probability is known as classical theory of probability.

Random Experiments :

An experiment is called random experiment i f it satisfies the following two conditions:

(i) It has more than one possible out come

(ii) It is not possible to predict the outcomes in advance.

Outcomes and sample space :

A possible result of a random experiment is called its outcomes. The set of outcomes is called the sample space of the experiment.

Event- Any subset E of a sample space S is called an event.

Mutually exclusive events:

In the experiment of rolling a die, a sample is S = {1, 2, 3, 4, 5, 6). consider events A 'an old number appears' and B 'an even number appears'.

Clearly the event A excludes the event B any vice versa. In other words, there is no outcomes which ensures the occurrence of events A and B simultaneously.

Here A = {1, 3, 5} and B = { 2, 4, 6}

Clearly $A \cap B = \phi$ i.e. A and B are disjoint Sets.

In general, two events A and B are called mutually exclusive events if the occurrence of any one of them excludes the occurrence of the other event i.e. If they can not occur , simultaneously.

Ex. Two dice are thrown and the sum of the numbers which come up on the dice is noted. Let us consider the following events associated with the experiment.

A : The sum is even

B : the sum is multiple of 3.

C : The sum is less then 4

D : the sum is greater than 11.

which pairs of theese events are mutually exclusive

Solution : There are 36 elements in the sample

Space S = {(x, y) : x, y = 1, 2, 3, 4, 5, 6} Then

A = {(1,1,), (1,3), (1,5), (2,2), (2,4), (2,6)

(3,1), (3,3), (3,5), (4,2), (4,4), (4,6)

(5,1), (5,3), (5,5), (6,2), (6,4), (6,6)}

B = {(1,2), (2,1), (1,5), (5,1), (3,3), (2,4),

(4,2), (3,6), (6,3), (4,5), (5,4), (6,6)}

C = {(1,1), (2,1), (1,2)} and D = {(6,6)}

we find that

$A \cap B$ = {(1,5), (2,4), (3,3), (4,2),

(5,1), (6,6)} $\neq \phi$

There A and B are not mutually exclusive. Similarly $A \cap C \neq \phi$,

$A \cap D \neq \phi$, $B \cap C \neq \phi$ and $B \cap D \neq \phi$

Thus the pairs of events (A, C), (A, D), (B,C), (B, D) are not mutually exclusive events.

Also $C \cap D = \phi$ and so C and D are mutually exclusive events.

Conditional Probability :

If we have two events from the same sample space, does the information about the occurrence of one of the events affect the probability of the other event? Let us try to answer this question by taking up a random experiment in which the outcome are equally likely to occur.

Consider the experiment of tossing three fair coins. The sample space of the experiment is

S = {HHH, HHT, HTH, THH, HTT, THT,

TTH, TTT}

since the coins are fair, we can assign the probability $\frac{1}{8}$ to each sample point. Let E be the event 'at least two heads appear' and F be the event 'first coin shows tail'.

Then E = {H HH, HHT, HTH, THH}

and F = { THH, THT, TTH, TTT}

Therefore P(E) = P ({HHH}) + P ({HHT}) + P ({HTH}) + P ({THH})

$$= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$$

and P(F) = P ({THH}) + P ({THT}) + P ({TTH}) + P ({TTT})

$$= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$$

Also $E \cap F$ = [{THH}]

with P ($E \cap F$) = P ({THH}) = $\frac{1}{8}$

Now, suppose we are given that the first coin throw fail i.e. F occurs, then what is the probability of occurrence of E? with the information of occurrence of F, we are sure that the cases in which the first coin does not result into a tail should not be considered while finding the probability of E.

This information reduces our sample space from the set S to its subset F for the event E.

Now, the sample point of F which is favourable to event E is THH

Thus, probability of E considering F as the sample space = $\frac{1}{4}$

This probability of the event E is called the conditional probability of E given that F has already occured and is denoted by P(E/F)

Thus P (E/F) = $\frac{1}{4}$

Note that the element of F which favour the event E are the common elements of E and F i.e. the sample points of E $\cap$ F.

Thus, we can also write the conditional probability of E given that F has occured as

$$P(E/F) = \frac{Number\ of\ elementory\ events\ favourable\ to\ E \cap F}{Number\ of\ elementry\ events\ which\ are\ favourable\ to\ F}$$

$$= \frac{n(E \cap F)}{n(F)}$$

dividing the numerator and denominator by total number of elementary events of the sample space we see that P(E/F) can also be written as

$$P(E/F) = \frac{\frac{n(E \cap F)}{n(S)}}{\frac{n(F)}{n(S)}} = \frac{P(E \cap F)}{P(F)}$$

Note - It is valid only when P(F) $\neq$ 0 i.e. F $\neq \phi$

Definition : If E and F are two events associated with the same sample space of a random experiment, the conditional probability of the event E given that F has occurred i.e.

$$P(E/F) = \frac{P(E \cap F)}{P(F)} \text{ provided } P(F) \neq 0$$

Properties of conditional Probability :

Let E and F be events of a sample space S of an experiment, thus we have

Property 1 : P(S/F) = P(F/F) = 1

we know that

$$P(S/F) = \frac{P(S \cap F)}{P(F)} = \frac{P(F)}{P(F)} = 1$$

Also $P(F/F) = \dfrac{P(F \cap F)}{P(F)} = \dfrac{P(F)}{P(F)} = 1$

Property 2 : If A and B are any two events of a sample space S and F is an event of S such that $P(F) \neq 0$, then

$P \{(A \cup B/F) = P(A/F) + P(B/F) - P(A \cap B)/F)\}$

In particular, if A and B are disjoint events

then $P \{(A \cup B)/F) = P(A/F) + P(B/F)$

We have

$$P\{(A \cup B)/F) \quad = \quad \frac{P[(A \cup B) \cap F]}{P(F)}$$

$$= \frac{P[(A \cap F) \cup (B \cap F)]}{P(F)}$$

$$= \frac{P(A \cap F) + P\,(B \cap F) - P\,(A \cap B \cap F)}{P(F)}$$

$$= \frac{P(A \cap F)}{P(F)} + \frac{P(B \cap F)}{P(F)} - \frac{P[(A \cap B) \cap F]}{P(F)}$$

$$= P(A/F) + P(B/F) - P[(A \cap B)/F$$

When A and B are disjoint events then

$$P\,(\{A \cap B)/F\}) \;=\; 0$$

or $P\{(A \cup B)/F) = P\,(A/F) + P(B/F)$

Property 3:

$P(E^1/F) = 1 - P\,(E/F)$

we know that $P\,(S/F) = 1$

- $P[(E \cup E^1)/F] = 1$
- $P(E/F) + F\,(E'/F) = 1$

Since E and E, are disjoint

- $P(E'/F) \;= 1 - P\,(E)F)$

Ex. If P(A) = $\frac{7}{13}$, $p(B) = \frac{9}{13}$ *and* $P(A \cap B) = \frac{4}{13}$

evaluate P(A/B)

Solution : we have $P(A/B) = \dfrac{P(A \cap B)}{P(B)}$

$$= \frac{4/13}{9/13}$$

$$= \frac{4}{9}$$

Ex. A family has two children what is the probability that both the children are boys given that atleast one of then is a boy.

Solution : Let b stand for boy and g for girl

The sample space of the experiment is

S = {(b, b), (g,b), (b, g), (g,g)}

Let E and F denote the following events

E : both the children are boys

F : Atleast one of the child is a boy

Thus E = {(b, b) and F = {(b, b), (g, b), (b, g)}

Now $E \cap F = \{(b,b)\}$

Thus P(F)= $\frac{3}{4}$ $and P(E \cap F) = \frac{1}{4}$

Therefore $P(E/F) = \dfrac{P(E \cap F)}{P(F)} = \dfrac{1/4}{3/4}$

$$= \frac{1}{3}$$

Ex. A die is thrown twice and sum of the numbers appearing is observed to be 6. What is the conditional probability that the number 4 has appeared at least once?

Solution : Let E be the event that 'number 4 appears atleast once' and F be the event that sum of the numbers appearing is 6'.

Then E = {(4,1), (4,2), (4,3), (4,4), (4,5), (4,6)

(1,4), (2,4), (3,4), (5,4), (6,4)}

and F = {1, 5), (2,4), (3,3), (4,2), (5,1)}

we have $P(E) = \dfrac{11}{36}$ *and* $P(F) = \dfrac{5}{36}$

and E $\cap$ F = { (2,4), (4,2)}

Therefore P(E$\cap$ F) = $\dfrac{2}{36}$

Hence the required probability

$$P(E/F) = \frac{P(E \cap F)}{P(F)} = \frac{\frac{2}{36}}{\frac{5}{36}} = \frac{2}{5}$$

Multiplication theorem as Probability :

Let E and F be two events associated with a sample S. The event $E \cap F$ is also written as EF. The probability of event EF is obtained by using the conditional probability as obtained below:

$$P(E/F) = \frac{P(E \cap F)}{P(F)}, \qquad P(F) \neq 0$$

We can write $P(E \cap F) = P(F).\ P(E/F)$ .... (i)

Also we know that

$$P(F/E) = \frac{P(F \cap E)}{P(E)}, \qquad P(E) \neq 0$$

or $P(F/E) = \dfrac{P(E \cap F)}{P(E)}$

Thus $P(E \cap F) = P(E).\ P(F/E)$ .... (ii)

Combining (i) and (ii), we find that

$P(E \cap F) = P(E).\ P(F/E)$

$= P(F).\ P(E/F)$. provided

$P(E) \neq 0$ and

$P(F) \neq 0$

Ex. An Urn contains 10 black and 5 white balls. Two balls are drawn from the Urn one after the other without replacement what is the probability that both drawn balls are Black?

Solution : Let E and F denote respectively the events that first and second ball drawn are black we have to find P(E$\cap$F) or (EF)

Now P(E) = P (Black ball in the first draw)

$$= \frac{10}{15}$$

Also given that the first ball draw is black i.e., Event E has occured, now there are 9 black balls and five white balls left in the urn. Therefore, the probability that the second ball draw is black, given that the ball in the first draw is black, is nothing but the conditioned probability of F given that E has occured.

$$P(F/E) = \frac{9}{14}$$

by multiplication rule of probability, we have

$$P(E \cap F) = P(E).\ P(F/E)$$

$$= \frac{10}{15} \; x \; \frac{9}{14} = \frac{3}{7}$$

Independent Events :

If E and F are two events such that the probability of occurrence of one of them is not affected by occurance of the other such events are called independent events.

Definition : Two events E and F are said to be independent if P(F/E) = P(F), provided P(E) $\neq$ 0

and P(E/F) = P/E), provided P(F) $\neq$ 0

Now by the multiplication rule of probability

we have $P(E \cap F) = P(E) . P(F/E)$

If E and F are independent then

$$P(E \cap F) = P(E) . P(F)$$

Ex. A die is thrown, if E is the event 'the number appearing is a multiple of 3' and F be the event 'the number appearing is even' then find whether E and F are independent?

Solution : We know that the sample space is

S = { 1, 2, 3, 4, 5, 6 }

Now E = {(3, 6)}, F = { 2, 4, 6} and $E \cap F$ = {6}

Then P(E) = $\frac{2}{6} = \frac{1}{3}$, $P(F) = \frac{3}{6} = \frac{1}{2}$ *and* $P(E \cap F) = \frac{1}{6}$

Clearly $P(E \cap F) = P(E) \cdot P(F)$

Hence E and F are independent events.

Partition of a sample space :

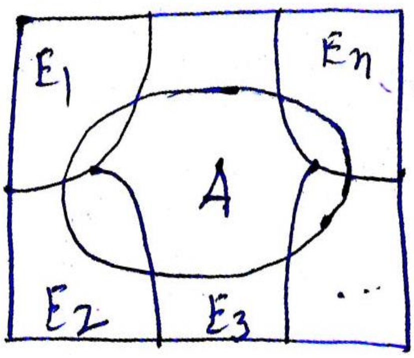
A set of events E1, E2 .... En is said to represent a partition of the sample space S if

(a) $Ei \cap Ej = \phi, \quad i \neq j, \; i, j = 1, 2, 3 ....n$

(b) $E1 \cup E2 \cup .... \cup En = S$ and

(c) $P(Ei) > 0$ for all $i = 1, 2, 3 ....n$

Theorem of Total Probability:



Let {E1, E2, ... En} be a partition of the sample space S, and Suppose that each of the events E1, E2, ..., En has non zero probability of occurence. Let A be any event associated with S, then

$P(A) = P(E1) \cdot P(A/E1) + P(E2) \cdot P(A) E2) + ...$

$+ P(En) \cdot P(A/En).$

$$= \sum_{j=1}^{n} P(Ej).\ \ P(A/Ej)$$

Proof : Given that E1, E2, ... En is a partition of the sample space S. Therefore

$$S = E1 \cup E2 \cup \ldots\ldots\ldots \cup En.$$

$$\text{and} \quad Ei \cap Ej = \phi, \ \ i \neq j, \ \ i, j = 1, 2, \ldots\ldots n$$

Now we know that for any event A

$$\begin{aligned} A &= A \cap S \\ &= A \cap (E1 \cup E2 \ldots\ldots\ldots \cup En) \\ &= (A \cap E1) \cup (A \cap E2) \ldots\ldots\ldots \cup (A \cap En) \end{aligned}$$

Also $A \cap Ei$ and $A \cap Ej$ are respectively the subsets of Ei and Ej. we know that Ei and Ej are disjoint for $i \neq j$, i, j = 1, 2, ... n.

$$\begin{aligned} \text{Thus } P(A) &= P[(A \cap E1) \cup (A \cap E2) \ldots\ldots\ldots\ldots \cup (A \cap En)] \\ &= P(A \cap E1) + P(A \cap E2) + \ldots\ldots + P(A \cap En) \end{aligned}$$

Now by multiplication rule of probability, have,

$$P(A \cap Ei) = P(Ei)\ .\ P(A/Ei) \text{ as } P(Ei) \neq 0$$

$$\forall \ \ i = 1, 2, \ldots\ldots\ldots n$$

Therefore

$$P(A) = P(E1) \, . \, P\,(A)\,E1) + ... + P(En) \, . \, P(A/En)$$

or $$P(A) = \, = \sum_{j=1}^{n} P(Ej) \, . \quad P(A/Ej)$$

Ex. A person has undertaken a construction job. The probabilities are 0.65 that there will be strike, 0.80 that the construction job will be completed on time if there is no strike, and 0.32 that the construction job will be completed on time if there is a strike Determine the probability that the construction job will be completed in time.

Solution : Let A be the event that the construction job will be completed on time, and B be the event that there will be a strike. We have to find P(A).

We have

$$P(B) = 0.65, \; P\,(\text{no strike}) = P(B^1) = 1\text{-}P(B)$$

$$= 1\text{-}0.65 = 0.35$$

$$P(A/B) = 0.32 \; , \; P\,(A/B^1) = 0.80$$

Since events B and B^1 form a partition of the sample space S, therefore, by theorem as total probability, we have

$$P(A) = P(B) \, . \, P(A/B) + P(B^1) \, . \, P\,(A/B^1)$$

$= 0.65 \times 0.32 + 0.35 \times 0.8$

$= 0.208 + 0.28 = 0.488$

Thus, the probability that the construction job will be completed in time is 0.488.

Bernoulli trails :

Definition :

Trials of a random experiment are called Bernoulli trails, if they satisfy the following conditions:

(i) There should be a finite number of trials.

(ii) The trails should be independent

(iii) Each trial has exactly two outcomes success or failure

(iv) The probability of success remains the same in each trial.

Binomial Distribution :

Consider the experiment of tossing a coin in which each trial results in success (say heads) or failure tails. Let S and F denote respectively success and failure in each trial. Suppose we are interested in finding the ways we have one success in six trials.

Clearly, six different cases are there as listed below.

SFFFFF, FSFFFF, FFSFFF, FFFSFF, FFFFSF, FFFFFS

Similarly, two successes and four failure can have $\frac{6!}{4! \, x \, 2!}$ combinations. It will be a lengthy job to list all of these ways. Therefore, calculation of probabilities of 0, 1, 2, ...n. number of successes may be lengthy and time consuming. to avoid the lengthy calculations and listing of all the possible cases, for the probabilities of number of successes in n- Bernoulli trials, a formula is derived For this purpose, let us take the experiment made up of three Bernoulli trials with probability p and q = 1 - p for success and failure respectively in each trial. The sample space of the experiment is the set.

S = {SSS, SSF, SFS, FSS, SFF, FSF, FFS, FFF}

The number of successes is a random variable X and can take values 0, 1, 2 or 3.

The probability distribution of the number of successes is as below :

P(X=0) = p (no success)

= P(FFF) = P(F) . P(F) . P(F)

= q - q q = q3 since the trials are independent

P(X=1) = P (one successes)

= P(SFF, FSF, FFS)

= P(SFF) + P (FSF) + P (FFS)

= P(S) P(F) P(F) + P(F) P(S) P(F)

+ P(F) P(F) P(S)

= p.q.q. + q.p.q. + qqp

= 3pq2

P(X=2) = p (two successes)

= P(SSF, SFS, FSS)

= P(SSF) + P(SFS) + P (FSS)

= P(S) . P(S). P(F) + P(S) P(F) P(S)

+ P(F) P(S) P(S)

= p.p.q + p.q.p + q.p.p.

= 3p2q

and P(X=3) = P (three successes) = P (SSS)

= P(S) . P(S) . P(S)

= p.p.p

= p3

Thus the probability distribution of X is

X	0	1	2	3
P(X)	q3	3q2p	3qp2	p3

Also the binomial expansion of (q+p)3 is

q3 + 3q2 p + 3qp2 + p3

Note that the probability of 0, 1, 2, or 3 successes are respectively the 1st, 2nd, 3rd and 4th terms in the expension of $(q+p)^3$.

Also since q+p = 1, it follows that the sums of these probabilities as expected is 1.

Thus we may conclude that in an experiment of n- Bernoulli trial, the probabilities of 0, 1, 2, ...n success can be obtained as 1st, 2nd .... $(n+1)^{th}$

terms in the expansion of $(q+p)^n$. thus

P(n successes) = ${}^nc_x\, p^x \quad q^{n-x}, \quad x = 0, 1, 2................n$

$q = 1-p$

P(X=x) is also denoted by

P(x) = ${}^nc_x \;\; q^{n-x}, \quad p^x, \; x = 0, 1,...................n,$

$q = 1-p$

This P(x) is called the probability function of the binomial distribution. the binomial distribution with n- Bernoulli trials and probabilities of success in each trial as p, is denoted by B (n, p).

Poisson Distribution

It is a descrete probability distribution. It is used in such cases where the value of p is very small i.e., p approaches zero ($p\to 0$) and the value of n is very large since in these cases binomial distribution does not give appropriate theretical frequencies, poisson distribution is found very appropriate. It is worth mentioning that a poisson distribution is a limiting form of Binomial Distribution as n moves towards infinity and p moves towards zero but np or mean remains constant and finite.

Poisson distribution is used to describe the behaviour of rare events such as number of germs in one drop of pure water, number of printing errors per page. Number of telephone calls arriving per minute at a telephone switch board etc.

Calculation procedure of Poisson Distribution :

Poisson distribution is a discrete distribution in which following steps are taken to find out the probabiliy of exactly 0, 1, 2, ...n successes :

(1) Firstly, we find out arithematic means of observed data, which is denoted as mi.e., $\overline{X} = m$

(2) Then the value of e^{-m} is obtained. The value of e = 2.7183, it is the base of the natural system of logarthims i.e.,

$$e^{-m} = \frac{1}{e^m} = \frac{1}{(2.7183)^m} = \frac{1}{A.L.(\log 2.7183 \ x \ m)}$$

$$= \frac{1}{A.L.(.4343 \ x \ m)}$$

(3) At that, the probabilities of 0, 1, 2, ...n successes are obtained by using the following formula of poisson

$$P(x) = e^{-m} \frac{m^x}{x!} \quad or \quad P(r) = e^{-m} \frac{m^r}{r!}$$

whereas X or Y = No. of successes 0, 1, 2, ...n

e = 2.7183

m = $\overline{X}$ = Arithmetic Mean

The probabilities of success would be as follows:

No. of Successes X	Probability p(X) or p(r)
0	e^{-m}
1	$m.e^{-m}$
2	$\frac{m2\ e-m}{2!}$
3	$\frac{m3\ e-m}{3!}$
4	$\frac{m4\ e^{-m}}{4!}$
r or X	$\frac{m^r.\ e^{-m}}{r!}$ or $\frac{m^x.\ e^{-m}}{x!}$

(4) Finally, expected frequencies are found out by multiplication of N (Total observed frequency) with the probabilities of success.

The value of Constants in Poisson distribution are as follows:

(i) Arithmetic Mean or $\overline{X} = m = np$

(ii0 Standard Diviation $\sigma = \sqrt{m} = \sqrt{np}$

(iii) Variance or σ^2 = np

(iv) $\mu_1 = 0$

(v) $\mu_2 = m$

(vi) $\mu_3 = m$

(vii) $\mu_4 = m + 3m^2$

(viii) $\beta_1 = \frac{\mu 3^2}{\mu 2^3} = \frac{m2}{m3} = \frac{1}{m}$

(ix) $\beta_2 = \frac{\mu 0}{\mu 2^2} = \frac{m+3m^2}{m^2} = 3 + \frac{1}{m}$

(x) $\gamma_1 = \sqrt{B1} = \frac{1}{\sqrt{m}}$

(xi) $\gamma_2 = \beta 2 - 3 = 3 + \frac{1}{m} - 3 = \frac{3}{m}$

Ex.1 In a radio manufacturing factory, average number of defective is 1 in 10 radios. Find the probability of a getting exactly 2 defective radios in a randem sample of 10 radios using (i) the binomial distribution

(ii) The poison distribution

Solution : According to binomial distribution

$$p = \frac{1}{10}, \quad q = 1 - \frac{1}{10} = \frac{9}{10}$$

Probability of 0, 1, 2, 3, ...n defective radios

$$= (q+p)^n = \left(\frac{9}{10} + \frac{1}{100}\right)^{10}$$

Probability of getting exactly 2 defective radios

$$^n p_r \,.\, pr\, q^{n-r} = 10_{C2} \left(\frac{1}{10}\right)^2 \left(\frac{9}{10}\right)^8$$

$$= \frac{10x9}{2x1} \; X \; \frac{1}{100} \; X \; \frac{4.3046721}{100000000}$$

= 45 x .01 x 4.305

= 0.193725

(ii) According to poison Distribution

$$p = \frac{1}{10}, \quad n = 10$$

$$m = np = \frac{10x1}{10} = 1$$

Probability of getting exactly 2 defective radios

$$p(2) = e^{-m} \,.\, \frac{m^2}{2!}$$

$$= e^{-1} \; x \; \frac{1^2}{2!}$$

$$= .3679 \; x \; \frac{1}{2}$$

$$= .18395$$

Ex.2. If the proportion of defective in a bulk is 4% Find the probability of not more than 2 defective in a sample of 10. It is known that e^{-4} = .6703.

Solution : Proportion of defective Units $p = \frac{4}{100} = .04$

$$m = np = .04 \text{ x } 10 = .4$$

Probability of not more than 2 defective means the sum of probabilities of 0, 1 and 2 defectives:

No. of defective Units	Probability
0 1 2	e^{-4} = 0.6703 $e-4 \; x \; \frac{4}{1} = .6703 \; x \; 4 = .2681$ $e^{-4} \; x \; \frac{.4^2}{2!} = \frac{.6703 \; x \; .4x.4}{2x1} = .0536$
Probability of defective upto 2 item	= .9920

Hence the probability of not more than 2 defectives

= 0.9920 or 99.2%

Normal Distribution :

Normal distribution is a continuous probability distribution in which the relative frequencies of a continuous variable are distributed according to normal probability law. In simple words, it is a symmatrical distribution in which the frequences are distributed evently about the mean of distribution.

The perfectly smooth and symmetrical curve, resulting from the expansion of the biomial $(p+q)^n$ when n approaches infinity is known as the normal curve. Thus the normal curve may be considered as the limit toward which the binomial distribution approaches as n increases to infinity. Alternatively we may say that the normal curve represents a continuous and infinite biomial distribution or simply a normal distribution.

Normal distribution is defined and given by the following probability density function:

$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} \quad e^{-\frac{1}{2}\left(\frac{X - \overline{X}}{\sigma}\right)}$$

$$-\infty < X < \infty$$

Where $\overline{X}$ = Mean, $\sigma =$ standard diviation and e = 3.1415 Normal distribution in its standard normal variate form is given by-

$$P(z) = \frac{1}{\sqrt{2z}} \quad e^{-\frac{1}{2}} z^2$$

$$-\infty < z \ \infty$$

where $z = \frac{\overline{X}\ \overline{X}}{\sigma}$

The mean of z is zero and standard deviation of z is 1.

Importance of Normal Distribution :

(i) Univarsality : this distribution is an universal distribution because except certain conditions almost in all areas nature of frequency distribution is normal.

(ii) Study of Natural Phenomenon- Almost all natural phenomenon possesses the feature of normal distribution such as height of adults, length of leaves of a tree etc. Therefore the normal distribution is widely used in the study of natural phenomenon.

(iii) Approximation is Bionomial and poison Distribution :

This normal distribution serves as a good approximation to Binomial and Poisson distribution particularly when she number of observations increases.

(iv) Basis of small samples- The whole theory of small sample is based on the fundamental assumption that the parent population from which the samples have been drawn follows normal distribution.

Area Under Normal Curve:

The equation of the normal curve depends on Mean ($\overline{X}$) and standard diviation (σ) and for different value of $\overline{X}$ and σ different normal curves are obtained. Since μ and σ can assume an infinite number of value it is impossible to tabulate the area under the curve for different values of μ and σ Therefore for the sake of convenience standard normal curve or unit normal curve is constructed with $\mu = 0$ and standard divation = 1 and thus the given value of the normal variate is transferred into standard units by the formula of z- trans-formation, as given below:

$$\overline{Z} = \frac{X - \overline{X}}{\sigma}$$

whereas Z = Z - transformation

$\overline{X}$ (or μ) = Arithmetic mean of population

X = value of observation

σ = S.D. of distribution

Ex. The mean of a distribution is 60 with standard diviation of 10. Assuming that the distribution is normal, what percentage of items be (i) between 60 and 72 (ii) between 50 and 60 (iii) beyond 72 and (iv) between 70 and 80.

Solution:

(i) Percentage of items between

60 and 72.

$$Z = \frac{X - \overline{X}}{S - D} = \frac{72 - 60}{10}$$

$$= 1.2$$

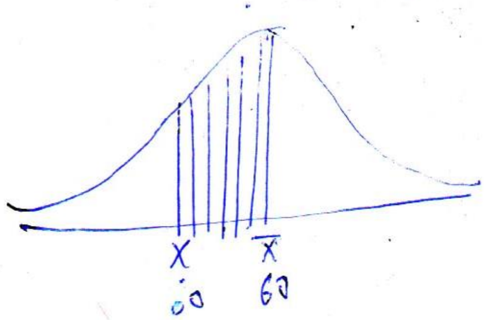
Area between 0 to 1.2 = .3849

0% of items between 60 and 72 = .3849 x 100 = 38.4%

(ii) Percentage of items between 50 to 60

$$Z = \frac{X - \overline{X}}{S.D}$$

$$= \frac{50 - 60}{10} = \frac{-10}{10} = -1$$



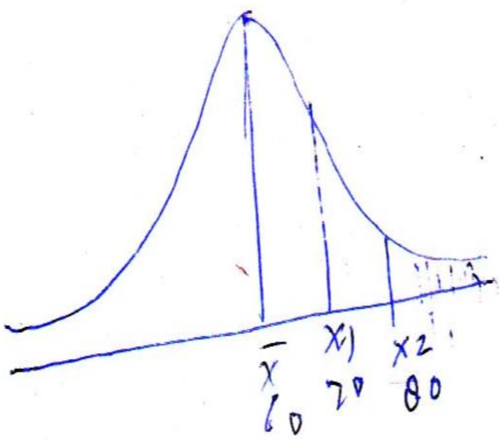
Shaded area = .3413

Area between 0 to -1 = .3413

0% of items between 50 and 60 = .3413 x 100 = 34.13%

(iii) percentage of items beyond 72

$$Z = \frac{72 - 60}{10} = \frac{12}{10} = 1.2$$



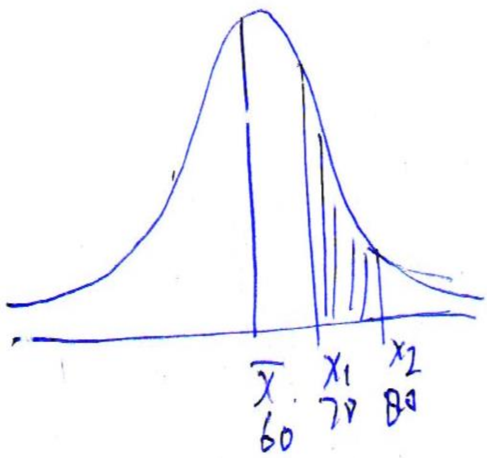
Area between 0 and 1.2 = .3849

area beyond 72 = .5000 - .3849

= 11.51

= 11.51%

(iv) between 70 and 80



$$Z_1 = \frac{X - \overline{X}}{S.D}$$

$$= \frac{70 - 60}{10} = \frac{10}{10} = +1$$

$$Z_2 = \frac{80 - 60}{10} = \frac{20}{10} = +2$$

Area from 0 to +1 = .3413

area from 0 to +2 = .4772

area between 70 and 80 = .4772 - .3413

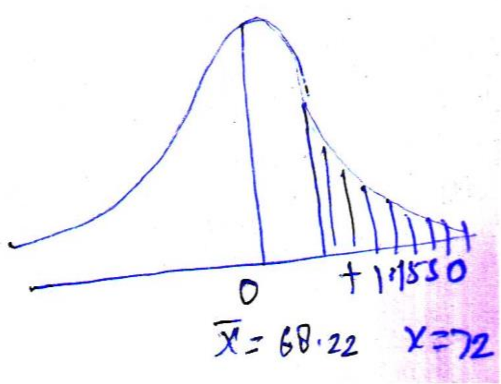
= .1359

= 13.59%

Ex.2. If the mean height of soldiers is 68.22" with a variance of 10.8". How many soldiers in a regiment of 1000 can be expected to be over 6 ft tall?

Solution - Given $\overline{X} = 68.22, \quad S.D^2 = 10.8$

$\overline{X} \;\; 6\,Feet \;\; or \;\; 72''.$



$$\overline{Z} = \frac{X - \overline{X}}{S\,.\,D} = \frac{72 - 68.22}{3.28}$$

$$= \frac{3.78}{3.28} = 1.15$$

Area between 0 and 1.15 = .3349

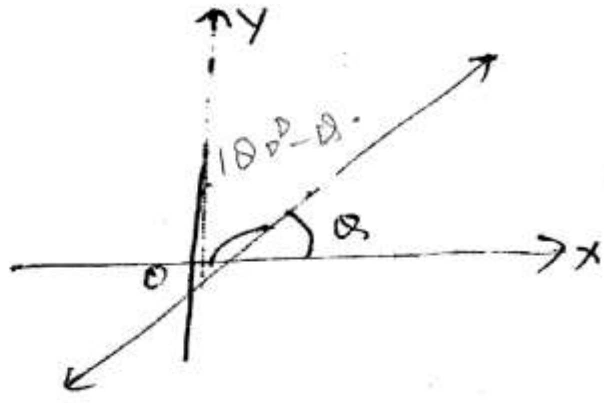
Probability of soldier being over 6 feet. ....

= .5000 - 3749 = .1251

No. of soldiers over 6 feet tall = 125

UNIT – 6 **Two Dimensional Geometry**

Slope of a line : A Line in a coordinate plane forms two angles with the x axis which are supplementary. The angle (say) θ made by the line l with positive direction of x-axis and measured anticlockwise is called the inclination of the line obviously $o^0 \leq \theta \leq 180^0$.



We observe that lines parallel to $x - axis$ or coinciding with $x - axis$, have inclination of o^0 . The inclination of a vertical line (parallel to or coinciding with $y - axis$) is 90^0 .

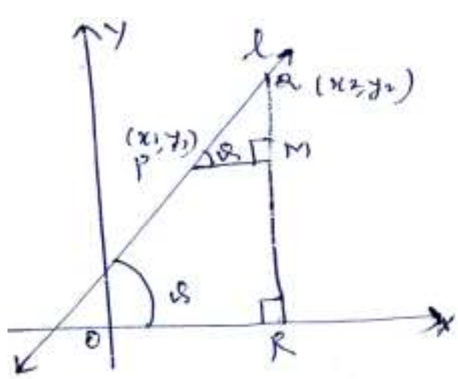
Definition : If θ is the inclination of a line l, then $tan\,\theta$ is called the slope or gradient of the line l. The slope of a line whose inclination is 90^0 is not defined.

The slope of a line is denoted by m. $Thus \quad m = \tan\theta,\ \theta \neq 90^0$.

It may be observed that the slope of $x - axis$ is zero and slope of $y - axis$ is not defined.

Slope of line when coordinates of any two points on the line are given : We know that a line is completely determined when we are given two points on it. Hence, we proceed to find the slope of a line in terms of the coordinates of two points on the line.

Let $P(x_1, y_1)$ and $Q\ (x_2, y_2)$ be two points on non-vertical lines l whose inclination is θ. Obviously $x_1 \neq x_2$, otherwise the line will become perpendicular to $x - axis$ and its slope will not be defined. The inclination of the line l may be acute or obtuse. Let us take two cases.



Draw perpendicular Q R to $x - axis$ and P M perpendicular to RQ as shown in above figure.

<u>Case I :</u> When angle θ is acute.

$$\angle MPQ = \theta$$

Therefore slope of line $l = m = tan\theta$

But in $\Delta MPQ,$ we have $tan\theta = \frac{MQ}{MP} = \frac{y_2 - y_1}{x_2 - x_1}$

We have $m = \frac{y_2 - y_1}{x_2 - x_1}$

<u>Case II :</u> When angle θ is obtuse, we have

$$\angle MPQ = 180^0 - \theta$$

Therefore $\theta = 180^0 - \angle MPQ$

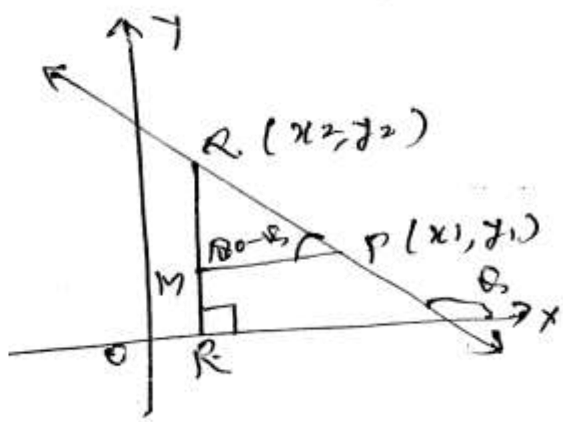
Now, slope of the line l

$$m = tan\theta$$

$$= (180^0 - \angle MPQ)$$

$$= -\tan LMPQ$$

$$= -\frac{MQ}{MP} = -\frac{y_{2} - y_{1}}{x_1 - x_2} = \frac{y_{2} - y_{1}}{x_2 - x_1}$$

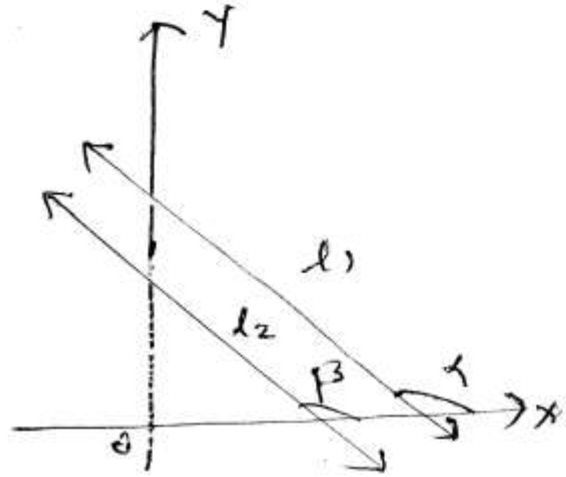


Consequently, we show that in both the cases the slope m of the line through the points (x_1, y_1) and (x_2, y_2) is given by $m = \frac{y_{2} - y_{1}}{x_2 - x_1}$

<u>Conditions for parallelism and Perpendicularity of lines in terms of their slopes:</u> In a coordinate plane, suppose that non-vertical lines l_1, and l_2 have slopes m_1 and m_2 respectively. Let their inclination be $\propto$ and β, respectively If the line l_1 is parallel to l_2. Then their inclination are equal $\propto = \beta$ hence $\tan \propto = tan\beta$ therefore $m_1 = m_{2,}$ their slopes are equal.

Conversely, if the slope of two lines l_1, and l_2 is same i.e $m_1 = m_2$

Then $tan \propto = tan\beta$



By the properly of tangent function (between 0^0 and 180^0) $\propto = \beta$. Therefore the lines are parallel. Hence, two non vertical lines l_1, and l_2 are parallel if and only if their slopes are equal.

<u>If the lines l_1 and l_2 are perpendicular :</u>

$$\beta = \propto + 90^0$$

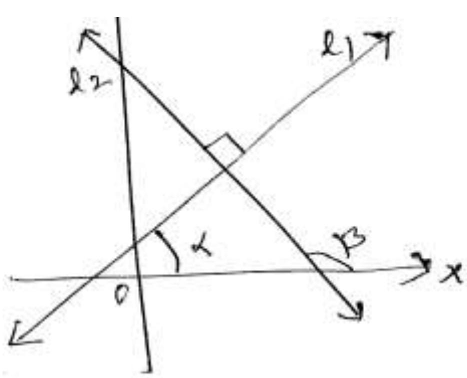
Therefore $tan\beta = \tan(\propto + 90)$

$$= -cot \propto = -\frac{1}{tan \propto}$$

Or $\quad m_2 = -\frac{1}{m_1}$ or $m_1 m_2 = -1$

Conversely, if $m_1 m_2 = -1$ $\quad$ i,e $tan \propto tan\beta = -1$

Then $tan \propto = -cot\beta = \tan(\beta + 90^0)$ or $\tan(\beta - 90^0)$



Therefore, $\propto$ and β differ by 90^0. Thus lines l_1 and l_2 are perpendicular to each other.

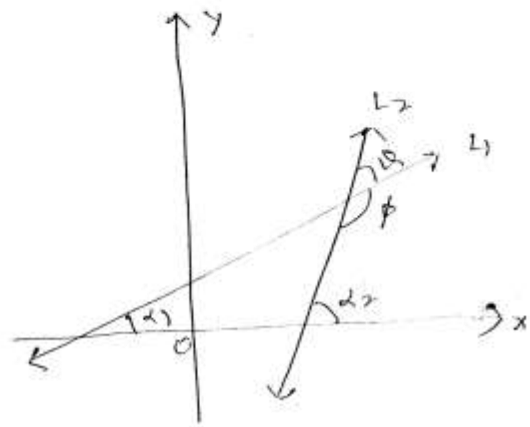
Hence, two non vertical line are perpendicular to each other if and only if their slope are negative reciprocals of each other

i.e., $\quad m_1 = -\frac{1}{m_2} \quad$ or $\quad m_1 m_2 = -1$

<u>Angle between two lines :</u> When we think about more than one line in a plane then we find that these lines are either intersecting or parallel.

Let L_1 and L_2 be two non-vertical lines with slopes m_1 and m_2 respectively. It $\propto_1$ and $\propto_2$ are the inclinations of line L_1 and L_2 respectively. Then

$$m_1 = tan \propto_1 \qquad \text{and} \qquad m_2 = tan \propto_2$$



We know that when two line interest each their, they make two pairs of vertically opposite angle such that sum of any two adjacent angle is 180^0. Let θ and $\emptyset$ be the adjacent angle between the lines L_1 and L_2.

Then $\qquad \theta = \propto_2 - \propto_1 \quad$ and $\quad \propto_{1,} \propto_2 \neq 90^0$

Therefore $\qquad tan\emptyset = \tan(\propto_2 - \propto_1) = \frac{tan \propto_2 - tan \propto_1}{1 + tan \propto_1 . tan \propto_2}$

$$= \frac{m_2 - m_1}{1 + m_2 . m_1}$$

$(as\ 1 + m_1 . m_2 \neq 0)$ and $\emptyset = 180^0 - \theta$ so that

$$tan\emptyset = \tan(180^0 - \theta) = -tan\theta = -\frac{m_2 - m_1}{1 + m_1 . m_2} \text{ as } 1 + m_1 . m_2 \neq 0$$

Now, there arise two cases:

Case I if $\frac{m_2 - m_1}{1 + m_1 . m_2}$ is positive then to $tan\theta$ will be positive and $tan\emptyset$ will be negative. Which means θ Will be acute and $\emptyset$ will be obtuse.

Case II if $\frac{m_2 - m_1}{1 + m_1 . m_2}$ is negative, the $tan\varphi$ will be negative and $tan\emptyset$ will be positive. Which means that θ will be obtuse and $\emptyset$ will be acute.

Then, the acute angle (say θ) between line $L_1 and\ L_2$ with slope $m_1 and\ m_2$ respectively is given by $tan\varphi = \left|\frac{m_2 - m_1}{1 + m_1 . m_2}\right|$, as $1 + m_1 . m_2 \neq 0$.

The obtuse angle (say ∅) can be found by using $\emptyset = 180^0 - \theta$

Ex.: If the angle between two line is $\frac{\overline{\Lambda}}{4}$ and slope of one of lines is $\frac{1}{2}$, find the slope of the other line.

Solution: We know that the acute angle θ between two lines with slopes m_1 and m_2 is given by $tan\theta = \left|\frac{m_2 - m_1}{1 + m_1.m_2}\right|$

Let $m_1 = \frac{1}{2}$, $m_2 = m$ and $\theta = \frac{\overline{\Lambda}}{4}$

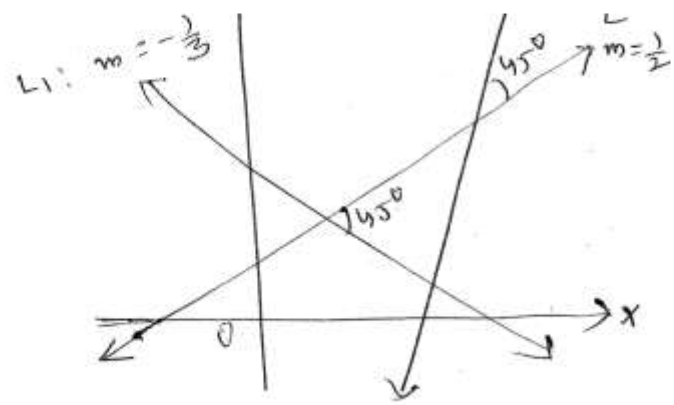
Now, putting these value in (i), we get

$$tan\frac{\overline{\Lambda}}{4} = \left|\frac{m - \frac{1}{2}}{1 + \frac{1}{2}m}\right| \text{ or } 1 = \left|\frac{m - \frac{1}{2}}{1 + \frac{1}{2}m}\right|$$

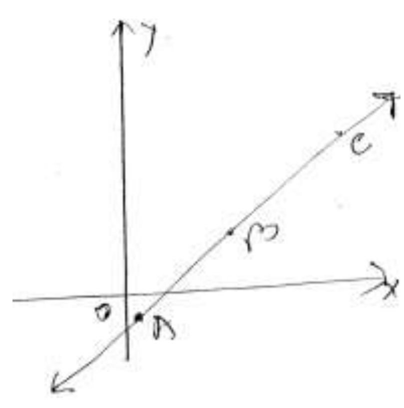
Which given $\frac{m - \frac{1}{2}}{1 + \frac{1}{2}m} = 1$ or $\frac{m - \frac{1}{2}}{1 + \frac{1}{2}m} = -1$

Therefore $m = 3$ or $m = -\frac{1}{3}$

Hence, slope of the other line is 3 $or -\frac{1}{3}$ figure explains the reason of two answer.

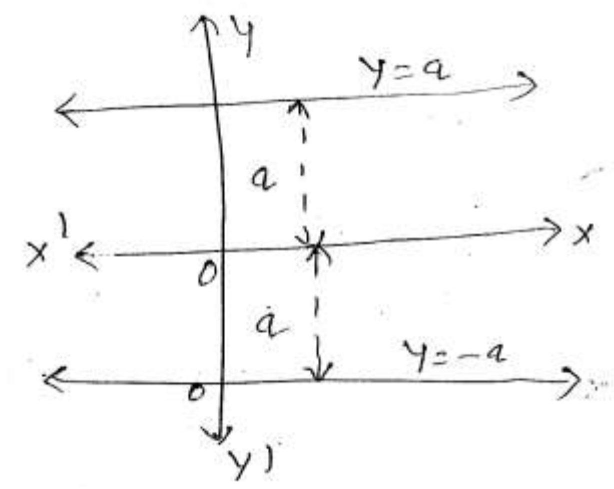


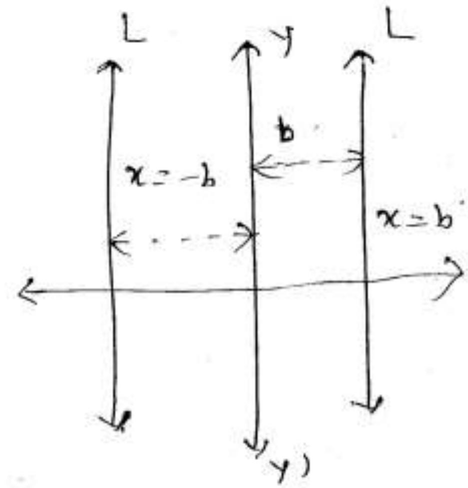
Collinearity of three points : We know that slopes of two parallel lines are equal. If two lines having the same slope pass thought a common point, then two line will coincide. Hence, if A, B and C are three points in the xy plane, then they will line on a line i, e, three points are collinear if and only if slope of $AB = slope$ of BC.



Various Forms of the Equation of a line :

Horizontal and Vertical Lines –

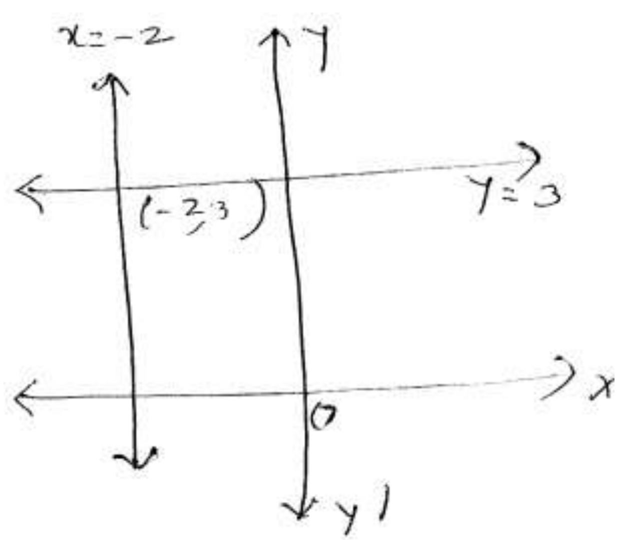




If a horizontal line L is at a distance a from the $X - axis$ then ordinate of every point lying on the line is either $a\ or - a$. Therefore, equation of the line L is either $y = a\ or\ y = -a$ choice of sign will depend upon the position of the line according as the line is above or below the $y - axis$. Similarly the equation of a vertical line at a distance b from the $y - axis$ is either $x = b\ or$ $x = -b$

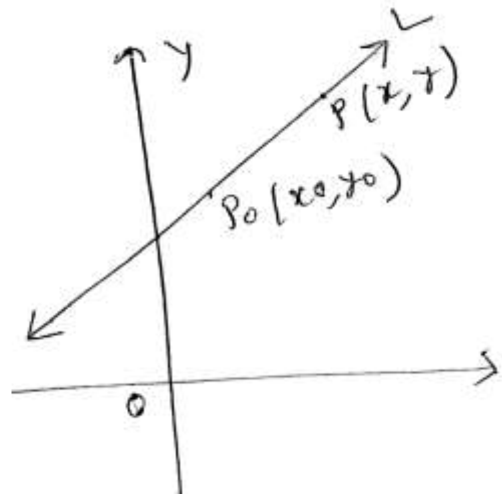
Ex.: Find the equation of the lines parallel to axes and passing though $(-2, 3)$

Solution: The Y coordinate of every point on the line parallel to $X - axis$ is 3, therefore, equation of the line parallel to $X - axis$ and passing through $(-2, 3)$ is $Y = 3$.



Similarly, equation of the line parallel to $Y - axis$ and passing through $(-2, 3)$ is $x = -2$

Point – Slope form: Suppose that $P_0(x_{0,}\ y_0)$ is a fixed point on a non-vertical line L, whose slope is m. $Lel\ P\ (x, y)$ be an arbitrary point on L.



Then by the definition, the slope of L is given by

$$m = \frac{y-y_0}{x-x_0} \quad i, e, \quad y - y_0 = m\ (x - x_0) \dots \dots (i)$$

Since the point $P_0(x_{0,}\ y_0)$ along with all points (x, y) on L satisfies (i) and no other point in the plane satisfies (i) equation (i) is indeed the equation for the given line L.

Then, the point (x, y) lies on the line with slope m through the fixed point $(x_{0,}\ y_0)$ if and only if, its coordinates satisfy the equation

$$y - y_0 = m\ (x - x_0)$$

Ex.: Find the equation of the line through $(-2, 3)$ with slope -4.

Solution: Here $m = -4$ and given point (x_0, y_0) is $(-2, 3)$ BY slope intercept form formula, equation of given line is

$$Y - 3 = -4(x + 2)$$

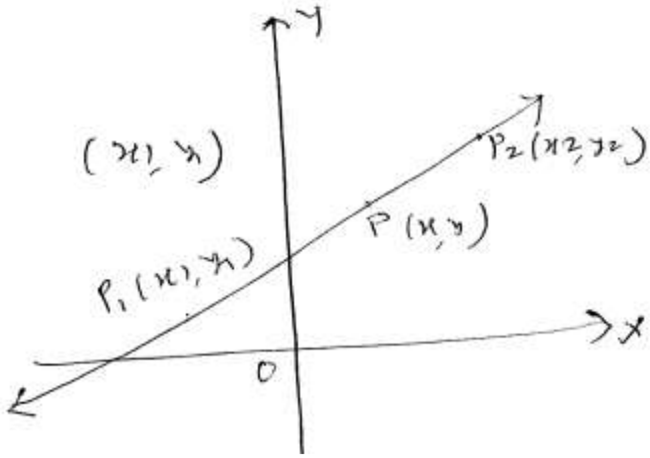
Or $\quad 4x + y + 5 = 0$ which is the required equation.

Two-Point form: Lel the line L passes through two given points $P_1\ (x_1, y_1)$ and $P_2\ (x_2, y_2)$. $Lel - P\ (x, y)$ be a general point on L.

The there points P_1 , P_2 and P are collinear, therefore we have slope of $P_1\ P =$

$slope\ of\ P_1 , P_2 \qquad i.e. \qquad \frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$

$$or \qquad y - y_1 = \frac{y_2-y_1}{x_2-x_1} \quad (x - x_1)$$



Thus equation of the line passing through the points (x_1, y_1) and (x_2, y_2) is given by

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}\ (x - x_1)$$

<u>Ex.:</u> Write the equation of the line through the point $(1, -1)$ and $(3,5)$.

<u>Solution:</u> Here $x_1 = 1,\ y_1 = -1,\ x_2 = 3,$ and $y_2 = 5$ using two point form for the equation of the line we have .

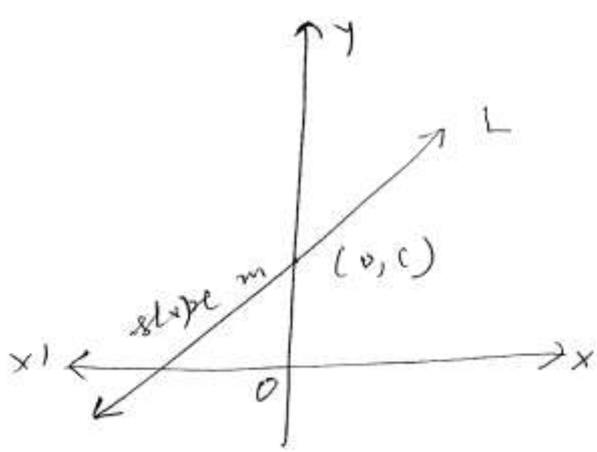
$$Y - (-1) = -\frac{5 - (-1)}{3-1}\ (x - 1)$$

Or $\quad -3x + y + 4 = 0 \quad$ which is the required equation.

<u>Slope-intercept form:</u> Sometime a line is known to us with its slope and an intercept on one of the axes. We will now find equation of such lines.

<u>Case I :</u> Suppose a line L with slope m cuts the $y - axis$ at a distance c from the origin. The distance c is called the y intercept of the line t. Obviously, coordinates of the point where the line meet the $y - axis$ are $(0, c)$. Thus L has slope m and passes through a fixed point $(0, c)$. therefore, by point slope form, the equation of L is

$$y - c = m\,(x - 0) \quad or \quad y = mx + c$$



Note that the value of c will be positive or negative according as the intercept is made on the positive or negative side of the $y-axis$, respectively.

Case II : Suppose a line L with slope m make $x-intercept$ d. Then equation of L is
$y = m(x-d)$

Ex. : Write the equation of the lines for which $tan\theta = \frac{1}{2}$, where θ is the inclination of the line and $y-intercept$ is $\frac{-3}{2}$ and $x-intercept$ is 4.

Solution : Here, slope of the line is $m = tan\theta = \frac{1}{2}$ and $y-intercept\ c = -\frac{3}{2}$. Therefore by slope intercept form, the equation of the line is

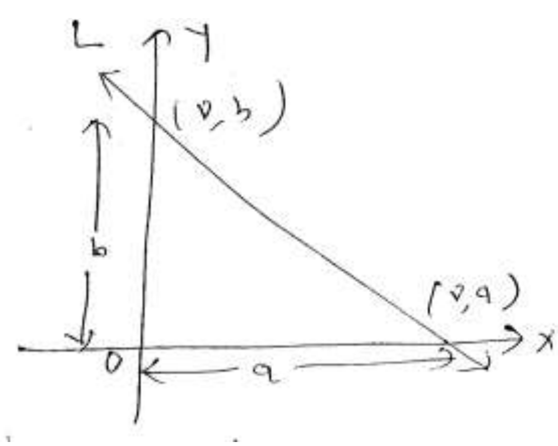
$$y = \frac{1}{2}x - \frac{3}{2} \quad or \quad 2y - x + 3 = 0$$

Again we have $m = tan\theta = \frac{1}{2}\ and\ \ d = 4$

Therefore by slope intercept form, the equation of the line is $y = \frac{1}{2}(x-4) \quad or \quad 2y - x + 4 = 0$ which is the required equation.

Intercept - form: Solution a line L make $x-intercept$ a and $y-intercept$ b on the axes. Obviously L meets $x-axis$ at the point $(a, 0)$ and $y-axis$ at the point $(0, b)$ BY two-point form of the equation of the line, we have $y - 0 = \frac{b-0}{\theta-a}(x-a) \quad or \quad ay = -bx + ab$

$$i,e \qquad \frac{x}{a} + \frac{y}{b} = 1$$



Thus, equation of the line making intercept a and b on x and y axis, respectively is

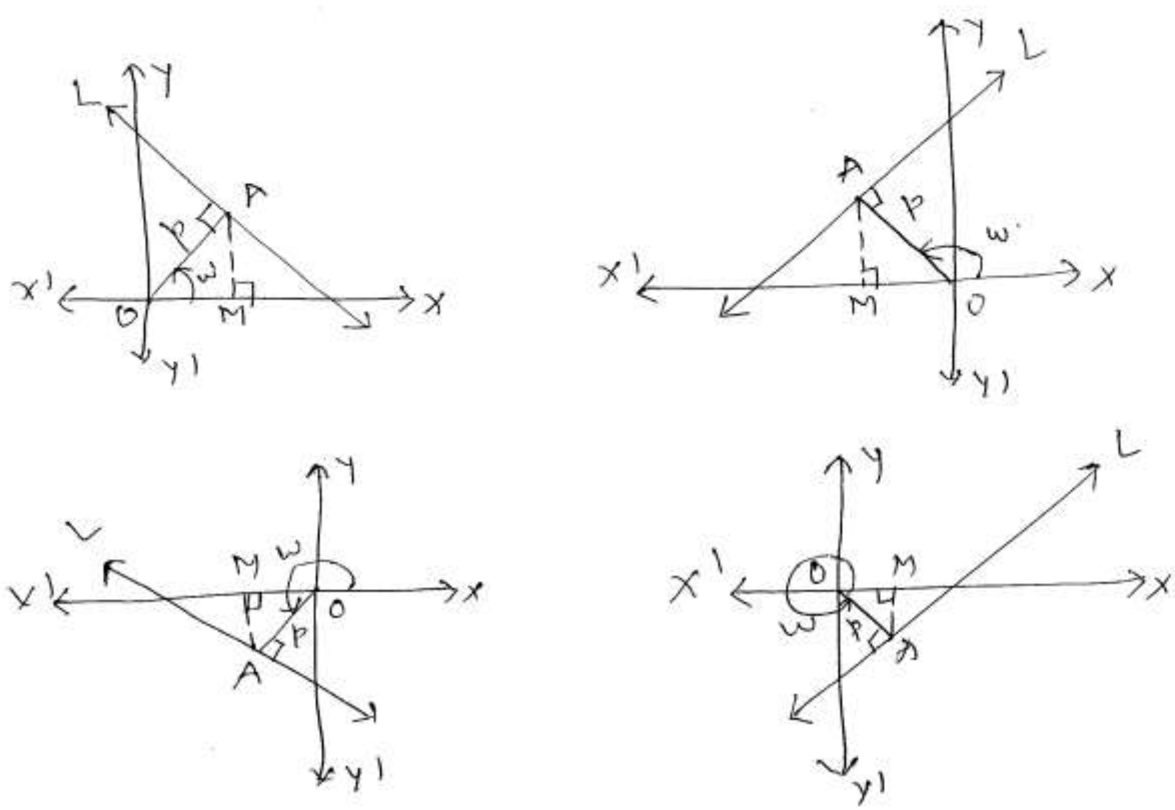
$$\frac{x}{a} + \frac{y}{b} = 1$$

Ex. : Find the equation of the line, which makes intercept -3 *and* 2 on the x *and* $y-axes$ respectively.

Solution : Here, $a=-3$ and $b=2,$ by intercept form to equation of the line is

$$\frac{x}{-3}+\frac{y}{2}=1 \qquad or \qquad 2x-3y+6=0$$

Normal - form :



Let L be the line, whose perpendicular distance from origin O be $OA=PX$. The possible position of line L in the Cartesian plane are shown in the above figure. Now our purpose is to find slope of L and a point on it. Draw perpendicular AM on the $x-axis$ in each case.

In each case, we have $OM = P\ coz\ W$ and $MA = PsinW$, so that the coordinates of the point A are $(P\ coz\ W, P\ sinW)$

Further, line L is perpendicular to OA. Therefore the slope of the line

$$L=-\frac{1}{slope\ \ of\ OA}=-\frac{1}{tanw}=-\frac{cosw}{sinw}$$

Thus, the line L has $slope -\frac{cosw}{sinw}$ and point $A\ (Pcosw, Psinw)$ on it. Therefore, by point-slope form the equation of the line L is $\qquad y-p\ sinw=-\frac{cosw}{sinw}(x-p\ cosw)$

Or $\qquad x\ cosw+y\ sinw=p(sin^2w+cos^2w)$

Or $\qquad x\ cosw+y\ sinw=p.$

Hence, the equation of the line having normal distance P from the origin and angle w which the normal makes with the positive directive of x-axis is given by $x\ cosw + y\ sinw = p$.

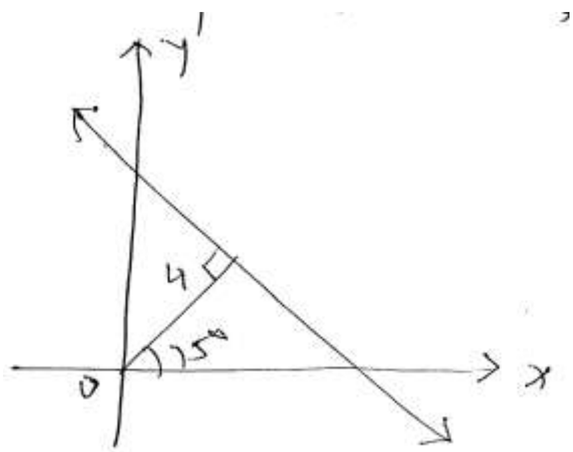
Ex. : Find the equation of the line, whose perpendicular distance from the origin is 4 units and the angle which the normal makes with positive direction of x-axis is 15^0.

Solution : Here, we are given

$$p = 4 \quad and \quad w = 15^0$$

Now $\cos 15^0 = \frac{\sqrt{3}+1}{2\sqrt{2}}$

and $\sin 15^0 = \frac{\sqrt{3}-1}{2\sqrt{2}}$



βy the normal form, the equation of the line is

$$x\ cosw15^0 + y\ sin15^0 = 4$$

or $\frac{\sqrt{3}+1}{2\sqrt{2}}\ x + \frac{\sqrt{3}-1}{2\sqrt{2}}\ y = 4$

or $\left(\sqrt{3}+1\right) x + \left(\sqrt{3}-1\right) y = \delta\sqrt{2}$

This is the required equation.

Exercise

1. The slope of a line is double of the slope of another line. If tangent of the angle between then is $\frac{1}{3}$, find the slopes of the lines.

2. If three points $(h, 0), (a, b)$ and (o, k) lie on a line, show then $\frac{a}{h} + \frac{b}{k} = 1$.

3. Find the equation of the line which safety the given condition.

   Passing through $(2, 2\sqrt{3})$ and inclined with the x-axis at an angle of 75^0

4. Perpendicular distance from the origin is 5 units and the angle made by the perpendicular with the position x-axis is 30^0.

5. A line perpendicular to the line segment joining the point (1,0) and (2,3) divide it in the ratio i:n find the equation of the line.

6. $P(a, b)$ is the mid-point of a line segment between axes. Show that equation of the line is

   $$\frac{x}{a} + \frac{y}{b} = 2.$$

7. BY using the concept of equation of a line, prove that the three points $(3,0), (-2, -2)$ and $((8, 2)$ are collinear.

UNIT -7 <u>Linear Programming</u> :- The Problem Which seek to maximize (or minimize) profit (or cost) form a general class of problems called optimization problems. Thus an optimization problem may involves finding maximum profit, minimum cost, or minimum use of resource etc.

A special but a very important class of optimization problem is linear programming problem.

<u>Linear programming problem and its mathematical formulation</u> :A furniture dealer deals in only two items - tables and chairs. He has Rs 50,000 to invest and has storage space of at must 60pieces. A table cost Rs 2500 and a chair Rs 500. He estimaty that from the sale of one table, he can make a profit of Rs 250 and that from the sale of our chair a profit of Rs 75. He wants to know how many tables our chairs he should by from the available many so as to maximize his total profit, assuming that he can sell all the items which is buys. In this Example we observe.

(i) The dealer can invest his money in buying tables or chairs or combination thereof. Further he would earn different profits by following different investments strategies.

(ii) There are certain overriding conditions or constraints viz, his investment is limited to a maximum of Rs 50000 and so his storage space which is for a maximum of go pieces.

Suppose he decides to buy tables only and no, so he can by 50000, 2500 i.e., 20 tables.

His profit in this case will be Rs (250x20) i.e., Rs 5000.

Suppose he chooses to buy chairs only and no tables with his capital of Rs. 50,000 he (an buy 50000, 500 i.e., 100 chairs. But he can store only 60 pieces. Therefor he is forces to buy only 60 chairs whice will give him a total profit of Rs (60x75) i.e., Rs 4500

There are many other possibilities for instance he may choose to buy 10 tables and 50 chairs, as he can store only 60 pieces. Total profit is this case would be Rs (10x250+50x75) i.e., Rs 6250 and so on.

We thus find that the dealer can invest his money in different ways and he would earn different profits bu following different investment strategics.

Now the problem is. How should he invest his money in order to get maximum profit ? To answer this question let us try to formulate the problem mathematically.

Mathematical formulation of the problem :

Let us be the number of tables and y be the number of chairs that the dealer buys obviously me and y mnet be non-negative i.e.,

$\mathcal{X} \geq 0$

$y \geq 0$

The chair is constrained by the maximum amount he can invest (Here it is Rs 50000) and by the maximum number of items he can store (Here it is 60) stated mathematically

2500x+500y£ 50000 (investment) constrains

or 5x + y £ 100

and x + y £ 60 (storage constraint) the dealer wants to invest in such a may so as to maximize his profit, say z which state as a function of x and y is given by z=25075y (called objective function) Mathematically the given problems now relative to maximize z = 250x + 75y

subject to the constraints :

5x + y £ 100

$\mathcal{X}$ + y £ 60

$\mathcal{X} \geq 0$ $y \geq 0$

So we have to maximize the linear function z subject to certain conditions determined by a set of linear inequalities with variable as non negative. Such problem are called linear programing problems.

Objective function : Linear function z=ax +by which a, b are constants, which has to be maximized or minimized is called a linear objective function.

Constraints : The linear inequalities or equation or restrictive on the variable at a linear programming problem are called constraints. The conditions $\mathcal{X} \geq 0$, $y \geq 0$ are called non-negative restriction.

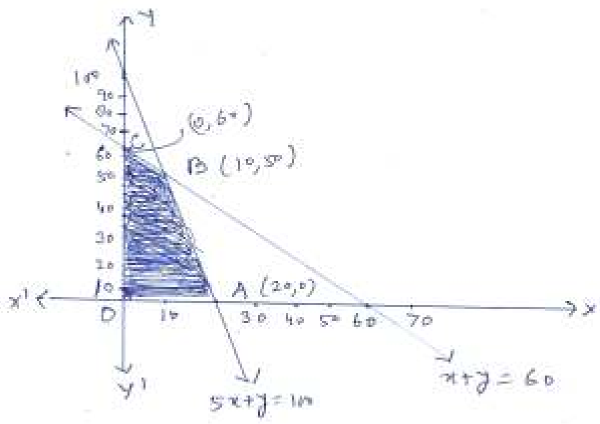
Optimization problem : A problem which such to maximize or minimize a linear function (say of two variable x and y) subject to a certain constraints as determined problem.

Different types of linear programming problems :

1 manufacturing problems - In this problems, we determine the number of units of different products which should be produced and sold by a firm when each product requires a fixed manpower machine hours, labour hour per unit of product, warehouse space per unit of the output etc, in order to make maximum profit.

2 Diet problems : In these problems we determine the amount of different kinds of constituent/nutrients which should be included in a diet so as to minimize the cost of the desired diet such that it contains a certain minimum amount of each constituent/nutrients.

<u>3 Transportation problem</u> : In these problems, we determine a transportation schedule in order to find the cheapest way of transporting a problem from plants/factaries situated at different locations to different markets.



let us refer to the problem of investment in tables and chairs. we will now solare the problem graphically. Let us graph the constraints stated as linear inequalities.

$$5x + y \leq 100 \text{ (i)}$$
$$x + y \leq 60 \text{ (ii)}$$
$$x \geq 0 \text{ (iii)}$$
$$y \geq 0 \text{ (iv)}$$

The graph of this system (shaded region) consists of the points common to all half planes determined by the inequalities is (i) to (iv). Each points in this region represents a feasible choice open to the dealer for investing in tables and chairs. The region, therefor is called the feasible region for the problem. Thus we have feasible region. The common regions determined by all the constraints including non-negative constraints x, y$\geq$0 of a linear programming problem is called the feasible region (as solution region) for the problem The region 0ABC is the feasible region for the problem the region other than feasible region is called an is feasible region.

Feasible solutions points within and on the boundary of the feasible region represent fesible solution of the constraints. In the about example the point (10,15) is a feasible solution of the problem and so are the points (0,60), (20,0) etc.

Any point outside the feasible region is called an infeasible solution. For example, the point (25, 40) is an infeasible solution of the problem.

Optimal (feasible) solution : Any points in the feasible region that gives the optimed values (maximum or minimum) of the objective function is called an optimed solution.

Now, we see that every point in the feasible region 0ABC satisfies all the constraints as given in (i) to (iv), and since these are infinitely many points, it is not evident how we should go about finding a point that gives a maximum value of the objective function z=250x+75y to handle this situation, we use the following theorems which are fundamental in soling linear programming problems.

Theorem : Let R be the feasible region (convex polygon) for a linear programming problem and let z>ax+by be the objective function when z has an optimed value (maximum or minimum) where the variable x and y are subject to constraints described by linear inequalities this optimed value much o ceur at a corner point (vertex) of the feasible region.

Theorem 2 : Let R be the feasible region for a linear programming problem and let z=ax+by be the objective function. If R is bounded them the objective function z has huth a maximum and a minimum value on R and each of these occurs at a corner point (vertex) of R.

Remark : Ir R is unbounded them a maximum or a minimum value of the objective function may not exist. However if it exists, it must occur at a corner point of R (by theorems)

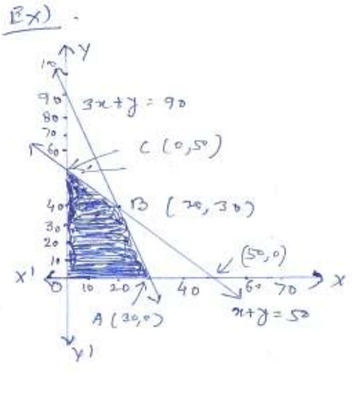
In the abour example the corner points (vertices) of the bounded (featible) region are : o, A, B, and C and it is easy to find their coordinates as (0,0) (20,0), (10,50) and (0,60) respectably. Let us how compate the value of z at these points

Vertex of the feasible region	Corresponding values of z (in Rs)
O (0,0)	0
C (0,60)	4500
B (10,50)	6250 Maximum
A (20, 0)	500

We observe that the maximum profit to the dealer results from the investments strategy (10,50) i.e., buying 10 tables and 50 chairs.

This method of solving linear programming problem is reformed as corner point method.

Ex) Solve the following linear programming problem graphically.



maximize z= 4x+y (i)

Subject to the constraints

$x+y \leq 50$ (ii)

$3x+y \leq 90$ (iii)

$x \geq o, y \geq o$ (iv)

Corner Point	Curresponding value of z
(0,0)	0
(30,0)	120 Maximum
(20,30)	110
(0,50)	50

The should region in the about liquors is the feasible region determind by the system of constraints (ii) to (iv) we observe that the feasible region OABC is bounded. So we hour use corner point method to determine the maximum value of z.

The co ordinators of the corner points O, A, B and C are (0,0), (30,0) (20,30) and (0,50) respectively. Num we absolute z at each corner point. Hens maximum value of z is 120 at the point (30,0)

Exercise

I

Solve the following linear programming problems graphically.

1 maximize $z= 3x4y$

Subject to the constraints :

$x+y \leq 4$, $x \geq 0$, $y \geq 0$

2 Minimize $z = -3x+xy$

Subject to $x+2y \leq 8$, $3x+2y \leq 12$, $x \geq 0$, $y \geq 0$

3 Maximize $z = 5x+3y$

Subject to $3x+5y \leq 15$, $5x+xy \leq 10$, $x \geq 0$, $y \geq 0$

Show that the minimum of z ocurs at more than two points.

4 minimize and maximize $z = 5 x+10y$

Subject to $x+2y \leq 120$, $x+y \geq 60$, $x-2y \geq 0$, $xy \geq 0$

5 Maximize $z = -x+2y$ subject to the constraints

$x \geq 3$, $x+y \geq 5$, $x+2y \geq 6$, $y \geq 0$

Unit - 8

Analysis of time based Data

(a) Index Numbers :

In its simplest form, an Index number is nothing more than a relative number, or 'relative' which expresses the relationship between two figures, where one of the figure is used as a base". Morris Hamburg.

Characteristics of Index Numbers :

1. **Expressed in Percentage-** Index number are expressed in terms of measure the extent of relative change. However, the percentage sign (%) is never used.
2. Absolute Number- Index numbers are free form units.
3. Index Numbers are specialised Averages- If two or more series are expressed in different units or they are composed of different types of item, their index number can be compared as they are specially designed for the purpose of comparison in situations where two or more series are expressed in different

units. Therefore, Index numbers are called specialized type of averages.

4. Index numbers are tools to measure relative change- Index numbers by their nature measure relative change in the values of a variable or a group of variables occur a period of time or between places.

USES OF INDEX NUMBERS :

1. Index number work as Economic Barometers:

Index numbers measure the pulse of economy of a country and measure the ups and downs in general economic condition of a country. Indices of prices, industrial production, agriculture production, foreign exchange, reserve bank deposits etc, determine the level of business activity of a country and these indices can be combined into a composite Index which could act as an economic barometer.

2. Index numbers help us in framing suitable policies:

Index number work as tools for the management of any organisation for efficient planning and formulation of business policies for example the increase in dearness allowances of the employees is decided on the basis of the cost of living Index.

3. Index numbers are helpful in determining trends and tendencies :

For example, by examining the Index number of Industrial production for the last few years, we can say about the trend of production whether it is increasing or decreasing.

4. Index numbers are used to measure the purchasing power of Money :

For example, the purchasing power of the Indian rupee in 1998 is only 10 paise as compared to its purchasing power in 1975. This means that a person who was having yearly income of Rs. 12000 in 1975 should have yearly income of Rs. 12000 in 1998 to maintain the same standard which he was maintaining in 1975.

PRICE INDEX NUMBER :

Price Index number measure changes in prices between two points of time and they are also used for the comparison of the prices of certain commodities Generally when we speak of Index number, it refers to price Index numbers.

Methods of Constructing Unweighted Index Numbers

(1) Simple Aggregative method :

In this method the total of current year prices for the various commodities is divided by the total of base year prices and the quotient is multiplied by 100 symbolically.

$$PO1 = \frac{\sum P1}{\sum PO} \ x \ 100$$

Where $\sum$P1 = Total of current year prices of various commodities

$\sum$P0 = Total of base y ear prices of various commodities

$\sum$Po1 = Current year Index Number

Ex.1 From the following data, construct price Index number for 1998 taking 1996 as the base year

Commodity	Price in 1996 (in Rs.)	Price in 1998 (in Rs.)
A	50	90
B	40	70
C	80	120
D	110	150
E	20	30

Solution : Construction of Price Index Number

Commodity	Price in 1996 (in Rs.)	Price in 1998 (in Rs.)
A	50	90
B	40	70
C	80	120
D	110	150
E	20	30
	$\sum P0 = 300$	$\sum P1 = 460$

From :

$$P01 = \frac{\sum P1}{\sum Po} \; x \; 100$$

$$= \frac{460}{300} \; x \; 100$$

$$= 153.33$$

It means that there is net increase in the price of commodities in the year 1998 in the extent of 53.33% as compared to 1996.

Ex.2: Compute the Index number for the years 1985 to 1992 by taking 1984 as the base year from the following data :

Year :	1984,	1985	1986,	1987,	1988,	1989,	1990,	1991,	1992
Price of X commodity	4	5	6	7	8	10	9	10	11

Solution : Construction of Index numbers taking 1984 as the base year.

Year	Price of Commodity X	Index Numbers
1984	4	100
1985	5	$\frac{5}{4}x100 = 125$
1986	6	$\frac{6}{4}x100 = 150$
1987	7	$\frac{7}{4}x100 = 175$

1988	8	$\frac{8}{4}x100 = 200$
1989	10	$\frac{10}{4}x100 = 250$
1990	9	$\frac{9}{4}x100 = 225$
1991	10	$\frac{10}{4}x100 = 250$
1992	11	$\frac{13}{4}x100 = 275$

(iii) Simple Average of Price relatives method:

A price relative is the number obtained by expressing the price for the current period as a percentage of the price of the base period.

Thus, if P0 and P1 denote the commodity price during the base period and the current period respectively, then

$$\text{Price relative} = \frac{P1}{P0}\ x\ 100$$

In the method of simple average of price relatives, first of all price relatives are computed for the various commodities and then average of these relatives is obtained by using any one of the measure of Central value i.e. AM or GM or HM or median or Mode, but generally either AM or GM are used to compute the average of price relatives.

If n is the number of items (commodities) in the list, then

$$Po1 = \left(\sum \frac{P1}{P0} \ x \ 100 \right) \quad \text{when AM is used}$$

Ex.: From the following data construct price Index number for 1997 taking 1995 as the base by simple aggregative method using arithmetic mean.

Commodities	Price in 1995 (in Rs.)	Price in 1997 (in Rs.)
A	50	70
B	40	60
C	80	90
D	110	120
E	20	20

Solution : Construction of Index number using AM of price relatives by simple aggregative method

Commodities	Price in 1997 (in Rs.)	Price in 1993 (in Rs.)	Price relatives $\frac{P1}{P0}\ x\ 100$
A	50	70	$\frac{70}{50}x100 = 140$
B	40	60	$\frac{60}{40}x100 = 150$
C	80	90	$\frac{90}{80}x100 = 112.50$
D	110	120	$\frac{120}{110}x100 = 109.10$
E	20	20	$\frac{20}{20}x100 = 100$
			$\sum\frac{P1}{Po}x100 = 611.6$

Now n = number of commodities = 5

From $P01 = \dfrac{\sum\frac{P1}{P0}\ x\ 100}{n}$

$= \dfrac{611.6}{5}$

= 122.32

It Shows that there is net increase in the prices of commodities in the year 1997 to the extent of 22.32% as compared to 1995.

Exercise

1. Construct Index number for 1994 with 1992 as base from the following prices of commodities by simple aggregative method.

Commodity	Price in 1992 (in Rs.)	Price in 1994 (in Rs.)
A	50	80
B	40	60
C	10	20
D	5	10
E	2	6

2. Construct the Index number for 1995 taking 1993 as base by price relative method using A.M.

Commodity	Price in 1995 (in Rs.)	Price in 1993 (in Rs.)
A	10	13
B	20	17
C	30	60
D	40	70

Time Series and trend analysis :

Meaning of Time series : Time series refers to such a series in which statistical data are presented on the basis of time of occurrence or in a chronological order. This measurement of time may be either year, month, week, day, hour or even minutes or seconds.

The following series is the example of Time series.

Year	Population (crores)
1961	43.9
1971	54.0
1981	68.4
1991	84.4
2001	102.7

Conclusively, it can be said that time series is an arrangement of statistical data in a Chronological order. For example, annual production of suger during ten years, population cencus after every ten years monthly price index numbers during a year et c.

Components of a Time Series

Time series is influenced collectively by a large variety of factors and forces. The effect of these forces can be classified in some definite categories. These categories are called the components of time series.

(i) Secular Trend or Long term Movement or Trend- Trend refer to that tendency which indicates the general direction of fluctuation in a long period.

It can be stated that despite various fluctuations from time to time, there will be an underlying tendency of movement in a particular direction and this tendency is called as long term trend. For example, that despite the fluctuations in prices in our country. The long term trend is of increasing. There are certain such facts also, in which tendency move to one direction only such as continuous increase in populations continuous decline in death rate etc.

The symbol of 'T' is used for denoting long term trend in the formulae relating to analysis of Time series.

(2) Regular Short time of Oscillations :

Most of the time series are influenced by such factors or forces which repeat themselves periodically. The variations arising out on account of such regular or periodical repetitions are called regular short time oscillations.

(3) Irregular or Random Fluctuations :

Irregular or random fluctuations occur accidently in time series. For instance, decline in profits due to broke of fire in the factory in a particular year, decrease in production due to sudden strike or scarecity of petroleum products due to war etc.

Analysis or Decomposition of Time Series - Meaning and Model:

Original data (0) given in time series include four components - (i) Trend or T (2) seasonal variations or S(3) cyclical Fluctuations or C (4) Irregular Fluctuations or I.

The measurement, analysis and study of these components is called analysis of time series. The measurement of four components of time series is based on models.

Additive Model :

This model is based on the assumption that the sum of four components is equal to original value i.e. 0 = T + S + C + I. This model assumes all components as residual, on the basis of which short - term fluctuations (S+C+I) can be found out by deducting trend (T) from original data (O) or O-T = S + C + I. similarly cyclical and irregular fluctuations can be found out by deducting seasonal variations from short term fluctuations i.e., O-T-S = C + I. If seasonal and cyclical fluctuation are isolated from short term fluctuations (O-T), irregular fluctuations can be measured i.e. O-T- (S+C) = O-T-S-C = I

Measurement of Trend-

Moving - average Method : Moving average method is a simple and flexible device of reducing fluctuations and obtaining trend values with a fair degree of accuracy. It consists in obtaining a series of moving averages (erythematic means) of successive overlapping groups or sections of the time series. For example, there are six years a, b, c, d, e and f and three years moving average is to be computed. It will be done as follows:

$$\frac{a+b+c}{3},\quad \frac{b+c+d}{3},\quad \frac{c+d+e}{3},\quad \frac{d+e+f}{3}$$

The basic question to be decided in this method is that what should be the period of moving average i.e. three yearly, four yearly, five yearly etc. This decision is taken on the basis of size of data and fluctuations therein. From the print of view of calculation of moving averages, the question can be divided in two categories- (1) When period is odd and (2) when period is even.

(i) Odd Period Moving Averages : It means moving average of odd period or years i.e., 3, 5, 7, 9, 11 ..... years. Its procedure can be explained as below on the assumption that three- yearly moving average are to be calculated-

(i) First of all, three yearly moving tools will be obtained. The total of first three years will be placed against the centre of three years i.e., second year.

(ii) After it, total of next three years (second, third and fourth) will be placed against third year, total if succeeding three years (third, fourth and fifth) will be placed against fourth year and this process will continue till the value of the last year i s included in the total.

(iii) Moving averages will be obtained by dividing each moving total by 3. It is important that moving averages will not be obtained for first and last year in case of 3 yearly moving averages and first two and last two year in case of 5 yearly moving averages.

Ex.1 From the following time series obtain trend value by 3 yearly moving averages.

Year	Sales (in 000 Rs.)	year	Sales (in 000 Rs.)
1998	8	2004	16
1999	12	2005	17
2000	10	2006	14
2001	13	2007	17
2002	15		
2003	12		

Solution : Calculation of trend values by three yearly moving average method.

Year	Sales (Thousand Rs.)	Three-yearly Moving Totals	Three-yearly Moving Average (Trend value)
1998	8	-	
1999	12	(8+12+10) = 30	10.00
2000	10	(12+10+13) = 35	11.67
2001	13	(10+13+15) = 38	12.67
2002	15	(13+15+12) = 40	13.33
2003	12	(15+12+16) = 43	14.33
2004	16	(12+16+17) = 45	15.00
2005	17	(16+17+14) = 47	15.67
2006	14	(17+14+17) = 48	16.00
2007	17		

Ex. 2 Calculate trend values from the following data assuming 5 yearly and 7 yearly moving average.

Year	1	2	3	4	5	6	7	8
Value	110	104	98	105	109	120	115	110
Year	9	10	11	12	13	14	15	16
Value	114	122	130	127	122	118	130	140

Solution : Calculation of Trend values by moving average method.

Year	**Value**	**Moving Tools**		**Moving Average**	
		5 Year	**7 Year**	**5 Year**	**5 Year**
1	110	-	-	-	-
2	104	-	-	-	-
3	98	526	-	105.2	-
4	105	536	761	107.2	108.71
5	109	547	761	109.4	108.71
6	120	559	771	111.8	110.14
7	115	568	795	113.6	113.57
8	110	581	820	116.2	117.14

9	114	591	838	118.2	119.71
10	122	603	840	120.6	120.00
11	130	615	843	123.0	120.43
12	127	619	863	123.8	123.29
13	122	627	889	125.4	127.00
14	118	637	-	127.4	-
15	130	-	-	-	-
16	140	-	-	-	-

(2) Even period moving Averages- If the moving average is to be calculated on the basis of even period i.e. 2, 4, 6 years, then average are calculated after centering the moving tools. Suppose, four- yearly moving tools are to be calculated, the following procedures would be adopted.

(i) First of all, four yearly moving tools will be obtained. The first total will be of first four years, the next total of four years excluding first year and this process will be repeated. The first total will be placed between second and third year, second total between third and fourth year and so on.

(ii) After it, these moving tools will be centred. For this purpose two period moving totals will be obtained.

(iii) Two period moving totals will be divided by 8.

Ex. From the following data calculate the 4 yearly moving averages and determine the trend values.

Year : 1998 1999 2000 2001 2002 2003 2004 2005 2006 2007

Value: 50.0 36.5 43.0 44.5 38.9 38.9 32.6 41.7 41.1 33.8

Solution : Calculation of Trend values by is four yearly Moving Averages

Year	Value	G-Yearly Moving Totals	2 Periods Moving Total Centred	Moving averages (Trend values)
1998 1999 2000 2001 2002	50.0 36.5 43.0 44.5 38.9	174.0 162.9 164.5	336.9 327.4 318.6	42.11 40.93

2003	38.1	154.1	305.4	39.83
2004	32.6	151.3	304.8	38.18
2005	41.7	153.5	302.7	38.10
2006	41.1	149.2		37.84
2007	33.8			

$$Moving \quad Average = \frac{2 Periods \quad Moving \quad Totals \quad Centred}{8}$$