केन्द्रीय माध्यमिक शिक्षा बोर्ड
CENTRAL BOARD OF SECONDARY EDUCATION

STUDENTS' HANDBOOK

APPLIED MATHEMATICS

SUBJECT CODE – 840
CLASS XI

DEPARTMENT OF SKILL EDUCATION

'SHIKSHA SADAN', 17, ROUSE AVENUE, INSTITUTIONAL AREA, NEW DELHI-110002

CONTENTS

APPLIED MATHEMATICS (CODE NO. 840)
SESSION 2019-2020

Syllabus of Applied Mathematics has been designed with an intention to orient the students towards the mathematical tools relevant in life. Special efforts has been made in order to connect it's application in various fields, so that, students who are opting for Social Science based subjects or Commerce based subjects or skill based subjects at senior secondary level can also fulfill their urge of learning mathematics joyfully.

OBJECTIVES:

a. To develop an understanding of basic mathematical and statistical tools and its application in Science, Business, Finance, Economics and other fields

b. To develop logical reasoning skills and enhancing problem solving abilities.

ASSESSMENT PATTERN FOR CLASSES XI & XII:

Theory	70 marks
Practical	30 marks
Total	**100 marks**

CLASS XI - SESSION: 2019-2020

UNIT	MARKS
1. Number Theory: **a. Prime Numbers:** Intersecting properties of prime number without proof, Ramanujan's work on Prime number, Encryption and prime number **b. Ratio, Proportion and Logarithms:** Business Application related to Ratio and Proportion. Practical Applications of Logarithms and Anti Logarithms	8
2. Interpretation of Data: Interpretation of Data represented in the form of charts, graphs, Frequency distribution, Histogram, Pie-chart etc.	8
3. Analysis of Data: Arithmetic Mean, Median, Mode, Geometric and Harmonic Mean, Range, Mean deviation, Standard Deviation, Variance, coefficient of variation, skewness.	10
4. Commercial Mathematics: Profit and Loss, Simple interest, compound interest, depreciation, Effective rate of interest, present value, net present value, future value, annuities.	12
5. Set Theory: Set and their representations, Empty set, Finite and Infinite sets, Equal sets, subsets, power set, universal set, Venn diagrams, union and intersections of sets, complement of set.	8
6. Relation and Function: Pictorial representation of a function, domain, co-domain and range of function, Function as special type of Relation, it's Domain and range.	5
7. Algebra: **a. Complex Number:** Concept of iota, imaginary numbers, arithmetic operation on complex number. **b. Sequence and Series:** Introduction of sequences, series, Arithmetic and Geometric Progression. Relationship between AM and GM, sum of n terms etc. **c. Permutations and Combinations:** Basic concepts of Permutations and Combinations, Factorial, permutations, results, combinations with standard results, Binomial Theorem (statement only).	14
8. Trigonometry: Trigonometric identities, calculation of Height and distance involving angles of all degrees till 90.	5
TOTAL MARKS (THEORY)	**70**

CLASS XII - SESSION: 2019-2020

UNIT	MARKS
1. Fundamentals of Calculus Basics of Limits & continuity, differentiation of non-trigonometric functions, Basic applications of derivatives in finding Marginal cost, Marginal Revenues etc. Increasing and Decreasing Functions, Maxima / Minima. Integration as reverse process of differentiation, integration of simple algebraic functions.	14
2. Algebra Introduction of Matrices, Algebra of Matrices, Determinants of Square matrices (Application only).	7
3. Logical Reasoning Number series, Coding, decoding and odd man out, direction tests, blood relations, syllogism, Binary numbers, logical operations and truth table.	8
4. Commercial Mathematics Calculating EMI, calculations of Returns, Compound annual growth rate (CAGR), Stocks, Shares, Debenture, valuation of Bonds, GST, Concept of Banking.	10
5. Probability Introduction to probability of an event, Mutually exclusive events, conditional probability, Law of Total probability. Basic application of Probability Distribution (Binomial Distribution, Poisson Distribution and Normal Distribution).	10
6. Two dimensional Geometry Slope of a line, equation of a line in point slope form, slope intercept form and two point form.	4
7. Linear Programming Introduction, related terminology such as constraints, objective function, optimization, different types of LP, mathematical formulation of LP problem, graphical method of solution for problems in two variables.	10
8. Analysis of time based Data **a. Index numbers:** meaning and uses of index number, construction of index numbers, construction of consumer price indices. **b. Time series & trend analysis:** Component of time series, additive models, Finding trend by moving average method.	7
TOTAL MARKS (THEORY)	**70**

SUGGESTIVE PROJECTS (FOR 30 MARKS)

- Algorithmic approach of Sieve of Erastosthene's.
- Ramanujan's theory of prime numbers: Use of prime numbers in coding and decoding of messages.
- Bertrnad's postulate
- Download http://pib.nic.in/prs/2011/latest31mar.pdf. Analyse various information that have been extracted from the Census, 2011. Understand as to how these information have been presented.
- Visit the census site of India http://www.censusindia.gov.in/Census_Data_2001/Census_Data_Online/Language/State_ment3.htm. Depict the information given there in a pictorial form.
- Prepare a questionnaire to collect information about money spent by your friends in a month on activities like traveling, movies, recharging of the mobiles, etc. and draw interesting conclusions.
- Check out the local newspaper and cut out examples of information depicted by graphs. Draw your own conclusions from the graph and compare it with the analysis given in the report.
- Analysis of population migration data – positive and negative influence on urbanization.
- Each day newspaper tells us about the maximum temperature, minimum temperature, and humidity. Collect the data for a period of 30 days and represent it graphically. Compare it with the data available for the same time period for the previous year.
- Draw a career graph of a cricketer (batting average for a batsman and bowling average for a bowler). Conclude the best year of his career. It may be extended for other players also – tennis, badminton, athlete.
- Share market data analysis – correlation and extreme fluctuation.
- Vehicle registration data – correlating with pollution and number of accidents.
- Visit a village near Delhi and collect data of various crops over past few years from the farmers. Also collect data about temperature variation and rain over the period for a particular crop. Try to find the effect of temperature and rain variations on various crops.
- How safe are privately owned public transport versus government owned public transport? Collect the data from archives about accidents of Blue Line buses and compare with those of DTC buses. Verify whether DTC buses are significantly safer.

- Visit Kirana shops near your home and collect the data of sale of certain commodities over a month. Try to figure out the stock of a particular commodity which should be in the store in order to maximize the profit.
- Mendelian Genetics: Genes are molecular units of heredity and carry certain information. They occur in pairs. Gregor Mendel studied about the inheritance in pea plants. One of the characteristics about their inheritance is smooth (S) and wrinkled (W). Suppose a plant has a heterogeneous gene that contains both the characteristic S and W. Find the probability of having a heterogeneous offspring (SW) or a homogeneous offspring (SS or WW) in the first generation, second generation, third generation, ….
- Choose any week of your ongoing session. Collect data for the past 10 – 15 years for the amount of rainfall received in Delhi during that week. Predict amount of rainfall for the current year.
- Stock price movement using the Binomial Distribution.
- Weather prediction (prediction of monsoon from past data).
- Risk assessments by insurance firms from data.
- Predicting stock market crash.
- Predicting outcome of election – exit polls.
- Predicting mortality of infants.
- Studying various games that use the concept of Probability – Lotto, Throwing dice, Khul Ja Sim Sim, etc.
- Data from iterative map for population growth – dynamics from plots.
- Statistical analysis of alphabets/ words appearing in a given text.
- Statistical methods for drug testing.
- Modelling population growth through data.
- Modelling spread of disease through data.
- Validation of existing models through data.

CENTRAL BOARD OF SECONDARY EDUCATION

STUDY MATERIAL: APPLIED MATHEMATICS {Class XI}

UNIT 1: (a) Number Theory
(b) Ratio, Proportion and Logarithms

UNIT – 1

(a) **Prime Numbers:** A prime number is a natural number greater than 1 that cannot be formal by multiplying two smaller natural numbers. For example, 5 is prime because the only ways of writing it as a product, 1 x 5 or 5 x 1.

Ramanujan prime are the last integers Rn for which there are atleast n prime between x and $\frac{x}{2}$ for all x $\geq$ Rn. The first Ramanujan prime are then 2, 11, 17, 29 and 41. Note that integer Rn is necessarily a prime number and hence, much increase by obtaining another prime at x = Rn.

(b) Ratio, Proportion and Logarithms: The value obtaining when two similar quantities are compared by dividing one quantity with the other is called ratio. The ratio of two quantities is that value which gives us how many times one quantity of the other. Only quantities of the same kind i.e. , the quantities with the same units can be compared.

NOTES:

1. The ratio a and b is written as a: b and is measured by the fraction $\frac{a}{b}$.
2. If two quantities are in the ratio a : b then the first and the second quantities will be $\frac{a}{a+b}$ times and $\frac{b}{a+b}$ times the sum of the two quantities respectively.
   Ex: If x: y =3:5, then $x = \frac{3}{8}$ (x+y) and $y = \frac{5}{8}$ (x+y).
3. The ratio of two quantities can be found , only when both the quantities are of the same kind .
   Ex: Ratio between 1 meter and 5 seconds cannot be found, as the quantities given are not of the same kind.
4. A ratio is an abstract quantity and a ratio does not have any units.
   Ex: The ratio of 30 seconds and one minute is 30 seconds: 60 seconds or 1:2.

Terms of a Ratio: For a given ratio a: b we say that is the first term or antecedent and b is second term or consequent.
In the ratio 3:4, 3 is the antecedent while 4 is the consequent.

Properties of a Ratio: The value of a ratio remain the same, if both the terms of the ratio are multiplied or dividedly same none. Zero quantity. If a, b and m are non-realnumber-/$\propto$

1. $\frac{a}{b}. = \frac{ma}{mb} \Longrightarrow$ a : b = am : bm.
2. $\frac{a}{b} = \frac{a/m}{b/m} \Longrightarrow a : b = \frac{a}{m} : \frac{b}{m}$.

Simplest form of a Ratio: The ratio of two or more quantities is said to be in the simplest form, if the highest common factor (H C F) of the quantities is 1. If the HCF of the quantities is not 1, then each quantity of the ratio is divided by the HCF to convert the ratio into its simplest form.

Ex: Express 81: 93 in its simplest form.

Solution: The HCF of 81 and 93 is 3 we divide each form by 3.

Thus $81:93 = \frac{81}{3} : \frac{93}{3} = 27:31$

$\therefore$ The ratio 81: 93 in its simplest form is 27 : 31.

Comparison of Ratio: Two ratios a: b and c: d can be compared in the following way; if $\frac{a}{b} > \frac{c}{d}$the: b >c: d.

Ex: Compare the ratios 4:5 and 18: 25.

Solution: Ratios can be also be compared by reducing then to equivalent fractions of a common denominator.

$4:5 = \frac{4}{5}$ and $18:25 = \frac{18}{25}$

$\frac{4}{5} = \frac{4X5}{5x5} = \frac{20}{25}$.

As $\frac{20}{25} > \frac{18}{25}$; $\frac{4}{5} > \frac{18}{25}$, $\therefore 4 : 5 > 18 : 25$.

Types of ratios:

1. A ratio a : b , where a > b. If a positive quantity x is added to the two terms in the ratio a : b , thus a + x : b + x < a : b.
2. A ratio a : b , where a < b , if a positive quantity x is added to the two terms of the ratio a : b, thus (a + x) : (b + x) > a : b.
3. A ratio a:b where a = b , if a positive quantity x is added to two terms in the ratio a : b thus (a + x) : (b + x) = a : b.
4. The compound ratio of two ratios a : b and c : d is ac : bd.

Ex: Divide Rs.> 80 among three friends P, Q and R in the ratio $\frac{1}{4} : \frac{1}{3} : \frac{1}{2}$,

Solution: Gives ratio is $\frac{1}{4} : \frac{1}{3} : \frac{1}{2}$,

The LCM of the denominators is 12

$\Longrightarrow \frac{1}{4}:\frac{1}{3}:\frac{1}{2} = \frac{1}{4} \text{ x } 12 : \frac{1}{3} \text{ x } 12 : \frac{1}{2} \text{ x } 12 = 3 : 4 : 6.$

The sum of the terms of the ratio = (3 + 4 + 6)= 13

P's share = Rs. $(> 80 \text{ x } \frac{3}{13})$ = Rs. 180.

Q's share = Rs. $(> 80 \text{ x } \frac{4}{13})$ = Rs. 240.

And R's share = Rs. $(> 80 \text{ x } \frac{6}{13})$ = Rs. 360.

Ex: if x: y = 3:4 and y : z = 6:11 , then find x :y : z.

Solution: In both the ratios y is common. Try to make the value of y equal in both the ratios.

Multiply each term of the ratio 6:11 by $\frac{2}{3}$ to make it first term (y) equal to 4.

$$\Longrightarrow y : z = 6 \text{ x } \frac{2}{3} : 11 \text{ x } \frac{2}{3} = 4 : \frac{22}{3}$$

$\therefore x : y : z = 3 : 4 : \frac{22}{3} = 9 : 12 : 22$

Ex: In a bag there are coins in the denominations Rs. ,Rs. 2 and Rs. 5 in the ratio 3 : 5 : 7, respectively

If the total value of the coins in the bag is Rs. 144, find the number of coins in each denomination and also the total value of Rs. 2 coins.

Solution: Let the number of coins of denominations of Rs. 1, Rs.2 and Rs. 5 be 3x, 5x and 7x, respectively. The total value of the coins in the bag:

= Rs.[3x + 5x (2) + 7 x (5)]

= Rs.(3xs + 10x + 35x)

= Rs. 48x

Given, the total amount in the bag = Rs. 144

$\Longrightarrow$ 48 x = 144

$\Longrightarrow$ x = 3

The number of Rs. 1 coins in the bag = 3x = 9

$\therefore$ The number of Rs. 2 coins in the bag = 5x = 5 (3) = 75

The number of Rs. 5 coins in the bag = 7x = 21.

∴ The total value of Rs. 2 coins in the bag =Rs. 15 x 2 = Rs.30.

PROPORTION: The equality of two ratios is called proportion.

Note: If a: b = c: d, then a, b, c and d are said to be in proportion and the same can be represented as a:b:: c : d.

Properties of Proportion:

1. a, b, c and d are respectively known as the first, second, third and fourth proportional.

2. The first and fourth terms are called extremes while the second and the third terms are called means.

3. Product of extremes = Product of means.

4. The fourth term can also be referred to as the fourth proportional of a, b and c.

5. If a : b = c : d, i.e., $\frac{a}{b} = \frac{c}{d}$ then the given properties can be written as b : a : : d : c i.e. $\frac{b}{a} = \frac{d}{c}$, by taking the reciprocals of terms on both sides. This relationship is known as 'invertened'.

6. If a : b : : c : d, then multiplying both sides of the proportion by $\frac{b}{c}$, we get a : c = b : d. This relationship is known as alternendo.

7. Adding 1 to both sides of the proportion

   a : b : : c : d, we get $\frac{a}{b} + 1 = \frac{c}{d} + 1$

   $\Rightarrow \frac{a+b}{b} = \frac{c+d}{d}$---------- (i)

   That is, (a + b) : b = (c + d) : d

   This relationship is known as 'componendo'.

8. Subtracting 1 from both sides of the proportion a : b : : c : d, we get.

   $\frac{a}{b} - 1 = \frac{c}{d} - 1 \Rightarrow \frac{a-b}{b} = \frac{c-d}{d}$---------- (ii)

   (a – b) : b = (c – d) : d.

   This relationship is known as 'dividendo'.

9. Dividing the eqn. (i) by eqn. (ii), we get

   $$\frac{\frac{a+b}{b}}{\frac{a-b}{b}} = \frac{\frac{c+d}{d}}{\frac{c-d}{d}}$$

$\Rightarrow$ (a + b) : (a : b) = (c + d) : (c – d)

This relationship is known as 'componendo and dividendo'.

10. If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$ and l, m and n are any three non-zero numbers, thus $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{la+mc+ne}{lb+md+nf}$.

Continued Proportion: Three quantities a, b and c are said to be in continued proportions if a : b :: b : c .

If a : b : : b : c, then c is called the third proportional of a and b.

Mean Proportional of a and c : If a : b : : b : c thus b is called the mean proportional of a and c.

We have already learnt that, product of means = Product of extremes.

$\Rightarrow$ b x b = a x c

$\Rightarrow b^2 = ac$

$\Rightarrow b = \pm\sqrt{ac}$

$\therefore$ The mean proportional of a and c is $\sqrt{ac}$.

Ex: If 4 taps can fill a tank in 10 hours, then in how many hours can 6 taps fill the same tank?

Solution: Assume that 6 taps can do the same work in x hour-/∝asmore taps require less hours to do the same work, number of taps vary inversely as the number of hour-/∝

$\therefore$ (Inverse ratio of number of taps) : : (Ratio of number of hours)

$\Rightarrow$6 : 4 : : 10 : x

$\Rightarrow \frac{6}{4} = \frac{10}{x}$

$\Rightarrow x = \frac{10x4}{6}$

$= 6\frac{2}{3}$ hour-/∝

$\therefore$ Six taps can fill the tank in $6\frac{2}{3}$hour-/∝

Ex: In a family, the consumption of power is 120 units for 18 days. Find how many units of power is consumed in 30 days.

Solution: Let the consumption of power is 30 days be x units.

The more the period of time, the more is the consumption of power. Hence the consumption of power varies directly as the number of days.

$\therefore$ (Ratio of number of days): : (Ratio of consumption of power)

$\Rightarrow 18 : 30 :: 120 : x$

Product of means = Product of extremes

$\Rightarrow 30 \text{ x } 120 = 18 \text{ x } \chi$

$\Rightarrow \chi = \frac{30 \text{ x } 120}{18} = 200$ units.

$\therefore$ The consumption of power for 30 days is 200 units.

Ex: 4 Men, each working 6 hours per day can build a wall in 9 days. How long will 6 men each working 3 hours per day take to finish the same work?

Solution: Clearly, $M \propto \frac{I}{D}$and $M \propto \frac{I}{H}$

$\Rightarrow$ MDI + = consort

$\Rightarrow M_1\ D_1\ I + I = Mz\ D_2\ H_2$

$\Rightarrow 4 \text{ x } 9 \text{ x } 6 = 6 \text{ x } D_2 \text{ x } 3$

$\Rightarrow D_2 = 12.$

$\therefore$ Six men each working 3 hours per day can finish the job in 12 days.

Logarithm: Logarithm is useful in long calculation involving multiplication and division.

Definition: The logarithm of any positive number to a given base (a positive number not equal to 1) is the index of the power of base which is equal to that number. If N and a ($\neq$1) are any two positive real numbers and for some real number x, $a^x = N$, thus x is said to be logarithm of N to the base a. It is written as $\log_a N = x$ i.e., if $a^x = N$, then $x = \log_a N$.

Ex: $2^3 = 8 \Rightarrow 3 = \log_2 8$.

From the definition of logs, we get the following results:

When $a > 0$, $b > 0$ and $b \neq 1$.

1. $\log_a a^n = n$
2. $a^{\log_a b} = b$.

System of logarithm: There are two systems of logarithm, natural logarithm and common logarithms, which are used most often.

1. Natural Logarithm: These were discovered by Napier. They are calculated to the based which approximately equal to 2.71.
2. Common Logarithms: Logarithms to the base 10 are known as common logarithms.

Properties:

1. Logs are defined only for positive real number-/∝
2. Logs are defined only for positive bases different from.
3. In$\log_b$ a,neither a nor b is negative but the value of $\log_b$ acan be negative.
   Ex: As$10^{-2} = 0.01$, $\log_{10} 0.01$ = -2
4. Logs of different numbers to the same base are different, i.e. if $a \neq b$, then $\log_m a \neq \log_m b$. In other words, if $\log_m a = \log_m b$, then $a = b$.
5. Logs of the same number to different bases have different values i.e. if $m \neq n$, then $\log_m a \neq \log_n a$. In other words, if
   $\log_m a = \log_n a$, then $m = n$.
6. Log of 1 to any base is 0.
7. Log of 0 is not defined.

Laws:

1. $\log_m(ab) = \log_m a + \log_m b$.
2. $\log_m(\frac{a}{b}) = \log_m a - \log_m b$.
3. $\text{Log } a^m = m \log a$.
4. $\log_b a \cdot \log_c b = \log_c a$
5. $\log_b a = \frac{\log_c a}{\log_c b}$.

   In this relation, if we take $a = c$, we get the following result:
   $\log_b a = \frac{1}{\log_a b}$.

   To find the log of a Number to base 10: Consider the following numbers: 2, 20, 200, 0.2 and 0.02.

   We setthat 20 = 10 (2) and 200= 100 (2)

   ∴Log 20 = I + log 2 and log 200 = 2 + log 2

   Similarly log 0.2 = -1+log2 and log 0.02 = -2+ log2 from the table, we see that log2 = 0.3010.

   ∴ Log20 = 1.3010, log200 = 2.3010, log 0.2 = -1+0.3010 and log 0.02 = -2+0.3010.

We note two points:

1. Multiplying or dividing by a power of 10 changes only the integral part of the log, not the fractional part.

2. For number less than 1. (for example 0.2) it is more convenient to learn the log value as -1+0.3010 instead of changing it to -0.6090.

For numbers less than 1, as the negative sign refers only to the integral
Part, it is written above the integral part, rather than in front i.e. , log 0.2 = T .3010 and not – 1.3010.

The integral part is called the characteristic and the fractional part (which is always positive) is called the mantissa.

Ex: Express -0.5229 in the standard form.

Solution: -0.5229 = -1+1-0.5229 = T.4771.

The rule to obtain the characteristics of log x.

1. If $x > 1$ and there are n digits in x, the characteristic is $n - 1$.
2. If $x < 1$ and there are m zeroes between the decimal point and the first non – zero digit of x , the characteristic is – m, more commonly written as $\overline{m}$.

Note: $-4 = \overline{4}$. But $-4.01 \neq \overline{4}.01$

To find the log of a number from the log tables: Find the value of log25, and log0.025.

Solution: In the log table, we find the number 25 in the left hand column. In this row, in the next column (under0), we find 0.3971.

For the log of all numbers whose significant digits are 25 and this number 0.3979, is the mantissa. Prefixing the characteristic we have

$\log 25 = 1.3979$
$\log 250 = 2.3979$
$\log 0.025 = \overline{2}3979$

Ex: Find the value of log 2.546 and log 25460.

Solution: In the log table we locate 25 in the first column. In this row, in the column under 4 we find 0.4048. Since there are four significant digits in 2546, in the same row where we have found 4048, under column 6 in the mean difference column, we find the number 10, the mantissa of the logarithm of 2546 will be $4048 + 10 = 4058$

Thus $\log 2.546 = 0.4058$
$\log 0.2546 = \bar{1}.4058$
$\log 25460 = 4.4058$

Antilog: As $\log_2 8 = 3$, 8 is the antilogarithm of 3 to the base 2. Antilog b to base m is m^{b}. We saw that, $\log 2.546 = 0.4058$. Therefore antilog $0.4058 = 2.546$.

Ex: to find the antilog of 2.3246.

Solution: we have to locate 0.32 in the left hand column and slide along the horizontal line and pick out the number in the vertical column headed by 4. We see that the number is 2109. The mean difference for 6 in the same line is 3. The significant digits in the required number are $= 2109 + 3 = 2112$. As the characteristic is 2, the required antilog is 211.2.

Ex: If $\log_{10} 3 = 0.4771$ and $\log_{10} 2 = 0.3010$, find the value of $\log_{10} 48$.

Solution: $\log_{10} 48 = \log_{10} 16 + \log_{10} 3$
$= \log_{10} 2^4 + \log_{10} 3$
$= 4\log_{10} 2 + \log_{10} 3$
$= 4(0.3010) - (0.477)$
$= 1.6811$

CENTRAL BOARD OF SECONDARY EDUCATION

STUDY MATERIAL: APPLIED MATHEMATICS {Class XI}

UNIT 2: INTERPRETATION OF DATA

UNIT- 2

INTERPRETATION OF DATA

To understand various trends of the data of a glance and to facilitate the comparison of various situations, the data are presented in the form of diagram and graphs.

There are a large number of diagrams which can be used for presentation of data. The solution of a particular diagram depends upon the nature of data. Objection of presentation, ability and experience.

Simple Bar Diagram: Simple bar diagrams comprises of group of rectangular bars of equal width for each class or category of data.

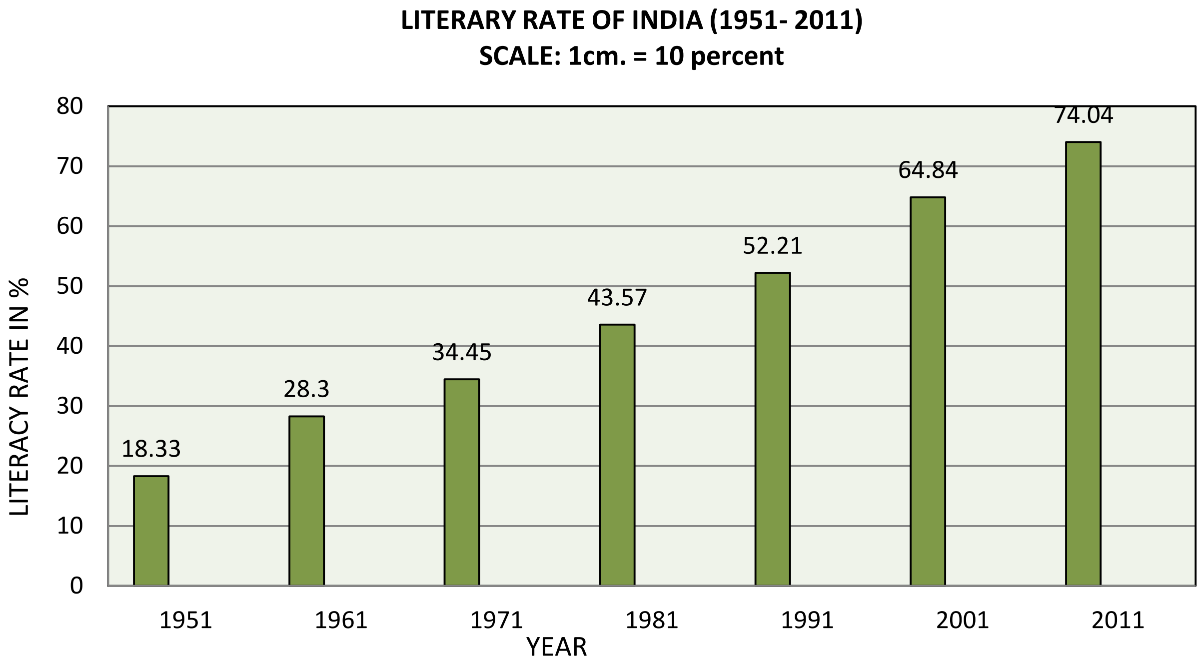
1. In this type of diagram, one bar represents only one figure. There will be as many bars as the number of figures.
2. These diagrams depict only one characteristics of the data.
3. The distance between the bars should be equal.
4. Height (or length) of the bar reads the magnitude of the data. The lower end of the bar touches the base line such that the height of a bar starts from the zero unit.
5. These diagrams can be either vertical or horizontal.
6. Bars of a bar diagram can be visually compared by their relation height and accordingly data are comprehended quickly.

Ex: The following table given data on literary rate in India. Present the information in the form of a simple (vertical) diagram.

Literary rate in india (1951 - 2011)

YEAR	1951	1961	1971	1981	1991	2001	2011
LITERARY RATE (IN PERCENTAGE)	18.33%	28.30%	34.45%	43.57%	52.21%	64.84%	74.04%

Solution:



PIE-DIAGRAM: Similar to sub – divided bar diagram a circle can also be partitioned into section to show properties of various components. Such a diagram is known as a pic diagram. The circle is divided into as many parts as there are compounds by drawing straight line from the centre to the circumference.

1. The size of the section indicates the proportion of each component part to the whole.
2. Different sectors representing various components parts should be distinguished from one another by using different shades, colors or by giving explanatory or descriptive labels.
3. Pie diagrams is also known as "Angular Circle Diagram".

Steps of Pie Diagram:

1. Construct a circle of an appropriate size with a compass considering the size of the paper and the data to be represented.
2. Convert the value of the various components into percentage value of the total.
3. The percentage value are converted in to corresponding degrees in the circle. Since one circle contains 360°. And percentage value of all the items are equal to 100, therefore our percentage value is represented by $\frac{360°}{100}$ = 3.6° to get size of angels.

4. Take a radians (preferably horizontally) as base line to draw the angle represented by first component. The new line will become the base for second components angular representation. Repeat this procedures till all the components are represented.
5. Represent various components by different shades, designs or colors for proper identifications.

Ex: Draw a pie-diagram to represent the following data of expenditure of an average working class family.

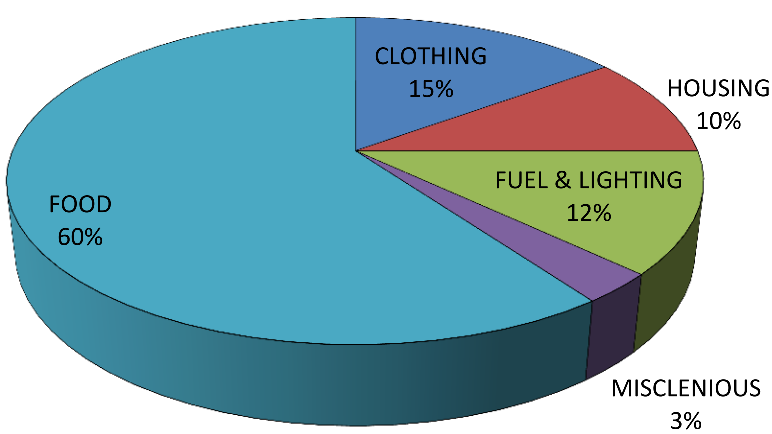
ITEMS OF EXPENDITURE	FOOD	CLOTHING	HOUSING	FUEL & LIGHTING	MISCLENIOUS
% OF TOTAL EXPENDITURE	60	15	10	12	3

Solution : The total of the percentage is 100. Hence , the total angle 360° represents 100. To find the proportionate angles, multiply each percentage by $\frac{360°}{100}$ or 3.6°.

These proportionate value are shown ahead:

ITEM OF EXPENDITURE	PERCENTGE(%) EXPENDITURE	PROPORTIONATE ANGLES
FOOD	60%	60 X 3.6 = 216°
CLOTHING	15%	15 X 3.6 = 54°
HOUSING	10%	10 X 3.6 = 36°
FUEL AND LIGHTING	12%	12 X 3.6 = 43.2°
MISCELLEANEOUS	3%	3 X 3.6 = 10.8°
	100%	360°

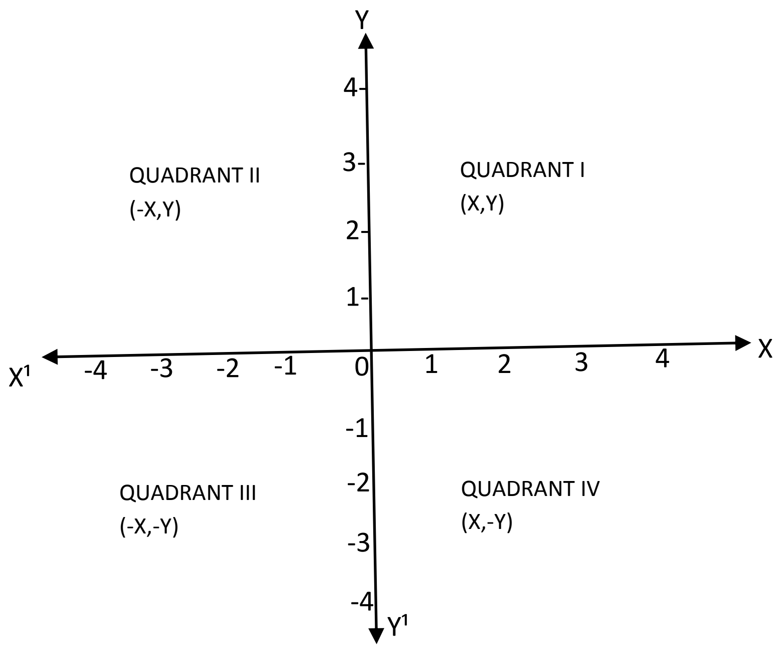
The circle is divided into 5 parts according to the degrees of angle at the centre.



Pie-diagram showing expenditure of an average working class family on different items.

Graphic Presentation: From the statistical point of view graphic presentation is much more accurate and appropriate than the diagrammatic presentation. Like diagrammatic presentation, graphic presentation also provides a quick and easier way of understanding various trends of data. It also facilitates the process of comparison of two more situations.

CONSTRUCTION OF GRAPHS: Graphs are generally drawn on a specially designed paper, known as Graph Paper. Graph paper is a fine network of horizontal and vertical lines, dividing every inch or centimeter into 10 equal parts.



How to construct a Graph:

Step 1 : In the first step draw two simple lines, which intersect each other as right angles. These lines are called axis.

Step 2 : The horizontal line is known as abscissa or x – axis and the vertical line as ordinate or y-axis.

Step 3 : The point at which they intersect each other is called point of origin, denoted by 0 represented by zero.

Step 4: By convention, the independent variable is normally measured along the x – axis and the dependent variable on the y- axis. The horizontal scale or the scale of x- axis need not begin with 0. The y –axis, however , should have a scale beginning with 0.

Step 5 : Select an appropriate scale, scale indicate the unit of a variable that a fixed length of axis would represent. It may be different for both the axis, but it should be taken in such a way so as to accommodate whole data on the given graph paper in a lucid and attractive style.

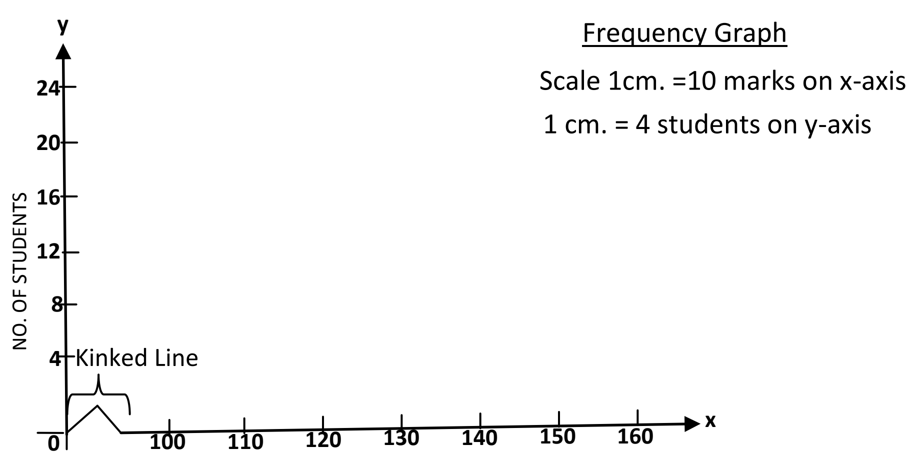
Step 6 : Plot the data on the graph papers on the basis of values of X and Y variable. It is essential to have two values : one representing variable X and the second representing variable Y, to plot a point on a graph.

Step 7 : Join the points (pair of values) on the graph paper.

Step 8 : Use index if different kinds of lines are drawn. Index should clearly show which line represents which variable.

Step 9 : Assign an appropriate title. This title should indicate the facts presented by the graph in a comprehensive and unambegious manner.

<u>Kinked Line</u> :



M

Y axis starting from zero, we can also start x-axis from zero ; In order to reduce the distance between 'zero' and the 'minimum value' on the horizontal axis (x-axis). We make use of 'kinked Line'. The Kinked Line is shown in the above figure.

<u>Frequency Distribution Graphs</u>: in case of frequency distribution, data is expressed in terms of an item or class – intervals. The class – limits or unit value are taken along the x-axis and frequencies along the y-axis.

<u>The most common forms of graphs of a frequency distribution are</u>:

1. Line frequency graph
2. Histogram
3. Frequency Polygon
4. Frequency curves
5. Ogive or cumulative frequency curve.

<u>Histogram</u>: Histogram is a graph of a frequency distribution in which the class intervals are plotted on the x-axis and their respective frequencies on the y-axis. It is a two- dimensional diagram.

a. Histogram is a very popular method of presenting continues series or class intervals frequency distributions. A histogram is never drawn for a discrete variable.
b. Height or length of each rectangle shows the frequency of the class and breadth indicates size of class – interval. Therefore, total area cover by Histogram represents the total frequencies.
c. Each rectangle is adjacent to other to give a continues picture.

<u>Equal class – interval are given</u> : In case of equal class intervals, the height of each rectangle is taken to be equal to the frequency of the corresponding class. In this case:

a. On the x-axis, each class interval is drawn, in which width of rectangle is equal to the magnitude of class interval.
b. On the y-axis, we plot the frequencies.

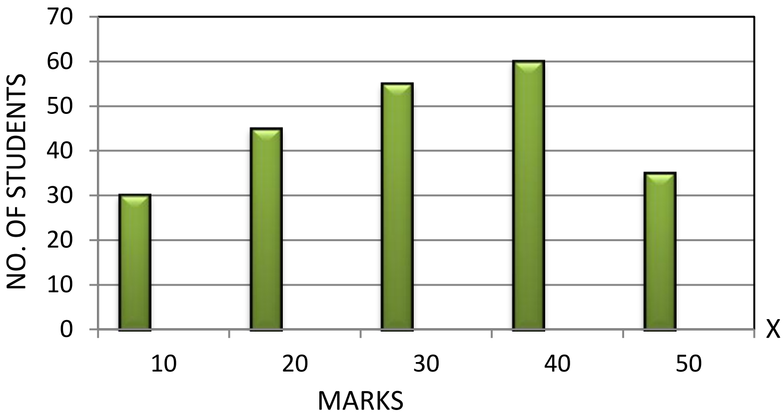
<u>Ex</u>: The frequency distribution of marks obtained by 225 students of a college is given below:

Draw a Histogram for the distribution

MARKS	0 – 10	10 – 20	20 – 30	30 – 40	40 – 50
NO. OF STUDENTS	30	45	55	60	35

Histogram showing Marks of students

Scale : 1 cm. = 10 marks on x-axis
1 cm. = 10 students on y-axis



Solution: for representing the above data by histogram, marks are plotted on the x-axis and number of students on the y-axis.

<u>Unequal Class – Intervals are given</u> : In case of unequal class – intervals, frequencies are first adjusted before constructing the histogram. The frequency distribution is adjusted in accordance to equal class width.

<u>Steps Needed to adjust frequencies</u>:

Step 1 : Determine width of lowest class – interval take this width as the standard one.

Step 2 : Determine width of other class – intervals and calculate 'Adjustment factor' for each class.

Adjustment Factor for any class = $\frac{Width\ of\ the\ class}{Width\ of\ the\ lowest\ class}$

Step 3 : The frequencies of the classes with the lowest width are not changed and frequencies of all other classes are adjusted by dividing it with the adjustment factor.

<u>Ex</u>: Prepare Histogram for the following data:

MARKS	0 -10	10 -20	20 -30	30 - 60	60 - 80	80 - 90	90 - 100
NO OF STUDENTS	5	10	4	18	4	3	9

Solution : In the given example, class intervals are unequal. Therefore, for preparing histogram, the frequencies are to be adjusted.

Step 1 : The width of lowest class – interval is 10.

Step 2 : Calculate Adjustment factor. It is to be calculated for 30 – 60 and 60 – 80.

Adjustment factor for 30 – 60 = $\frac{30}{10}$ = 3

Adjustment Factor for 60 – 80 = $\frac{20}{10}$ = 2

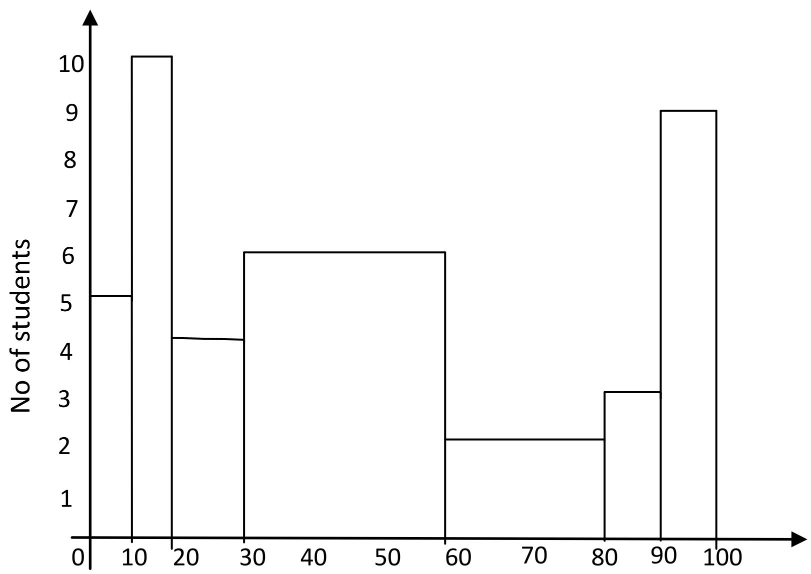
Step 3 : The width of 30 – 60 is 30, which is thrice in comparison to the lowest class interval. So, its frequency will be divided by 3. Similarly, width of 60 – 80 is 2 times more than the minimum class-interval. Therefore, frequency of this class will be divided by 2.

This histogram for adjusted frequency is shown as under:

MARKS	NO. OF STUDENTS	ADJUSTED FREQUENCY FOR HISTOGRAM
0 – 10	5	5
10 – 20	10	10
20 – 30	4	4
30 – 60	18	18 ÷ 3 = 6
60 – 80	4	4 ÷ 2 = 2
80 – 90	3	3
90 – 100	9	9

Histogram showing marks of students

Scale : 1cm. = 10 marks on x-axis.
1cm. = 1 student on y-axis.



Frequency Polygon: A frequency polygon is another method of representing a frequency distribution on a graph. Frequency polygon is an alternative to histogram and is also divided from histogram it self.

Frequency Polygon in Discrete series: In case of discrete frequency distribution, the following steps are followed:

Step 1 : Take the value of the variable on the x-axis and the corresponding frequencies on the y-axis.

Step 2 : Plot the various frequencies and the points are joined by a straight line.

Step 3 : The figure so obtained is extended to base (baseline) at both ends by joining the extreme points (first and last point) to the two hypothetical value of the variable before the first value of the variable and after the last value of the variable assume to have zero frequencies.

Ex: The following data shows number of rooms in 64 houses of a colony. Construct a frequency polygon.

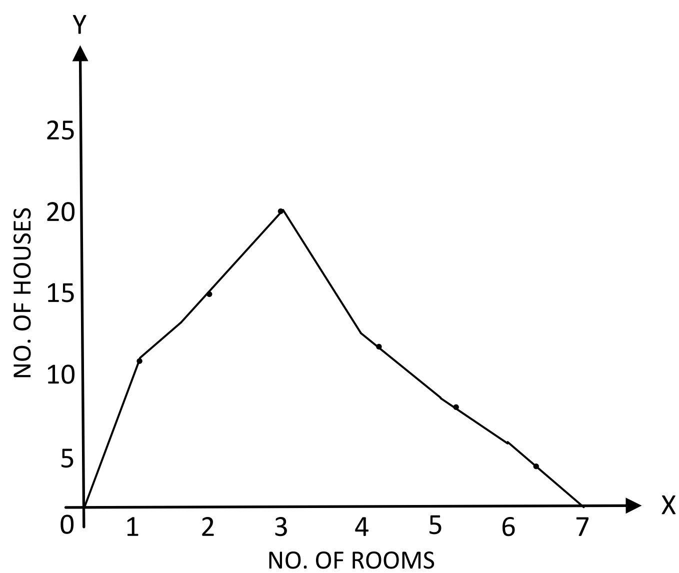
NO. OF ROOMS	1	2	3	4	5	6
NO. OF HOUSES	10	15	20	10	5	4

Solution : This is a case of discrete frequency distribution. So we will plot the number of rooms on the x-axis and the number of houses on the y-axis. The points obtained are joined by straight line giving a frequency polygon.

Frequency polygon showing number of rooms in Houses

Scale : 1 cm.: 1 Room on x-axis

1 cm. : 5 Houses y-axis



Frequency Polygon in Continues Series: In case of continues frequency distribution, there are two ways of making frequency polygon:

(i) With the help of Histogram
(ii) Without help of Histogram.

Frequency polygon with the help of Histogram: The frequency polygon through Histogram requires the following steps:

Step 1 : Draw the Histogram of the given frequency distribution.

Step 2 : Join the mid-point of the tops (upper horizontal sides) of the adjacent rectangles of the histogram by straight lines.

Step 3 : The figure so obtained is closed by extending the two ends to the base line. In doing this, two hypothetical classes of each end would have to be included, each with a zero frequency.

Ex: The following is the age distribution of workers of a factory. On the basis of this information, construct a histogram and convert it into a frequency polygon.

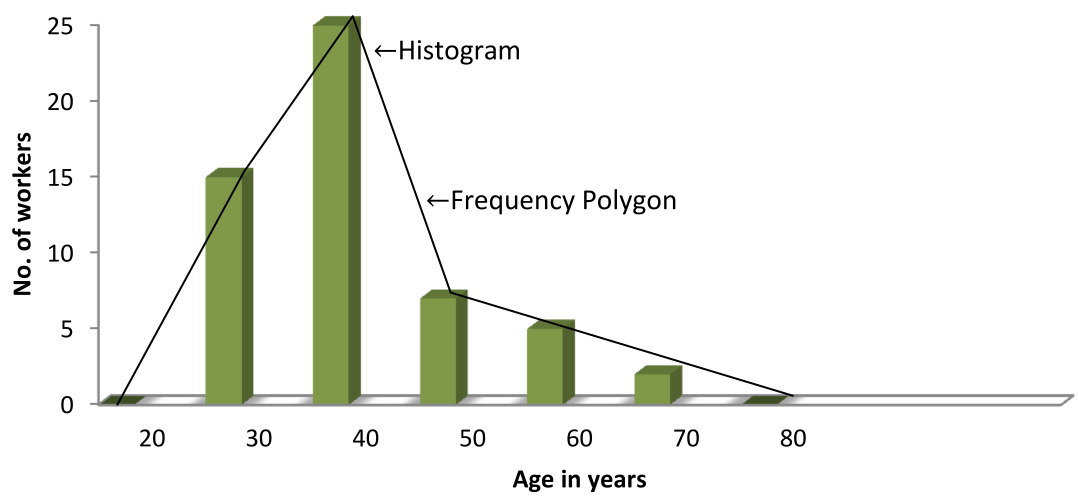
Age (in years)	20 – 30	30 – 40	40 – 50	50 – 60	60 – 70
No. of workers	15	25	7	5	2

Frequency Polygon (with histogram) showing age of workers

Scale : 1 cm. = 10 years on x-axis

1 cm. = 5 workers on y-axis

Solution :



(I) Frequency polygon without the help of histogram: The frequency polygon without Histogram requires the following steps:

Step 1 : Take mid-point of class-intervals on the x-axis and the frequencies on the y-axis.

Step 2 : Plot the various frequencies and the point are joined by a straight line.

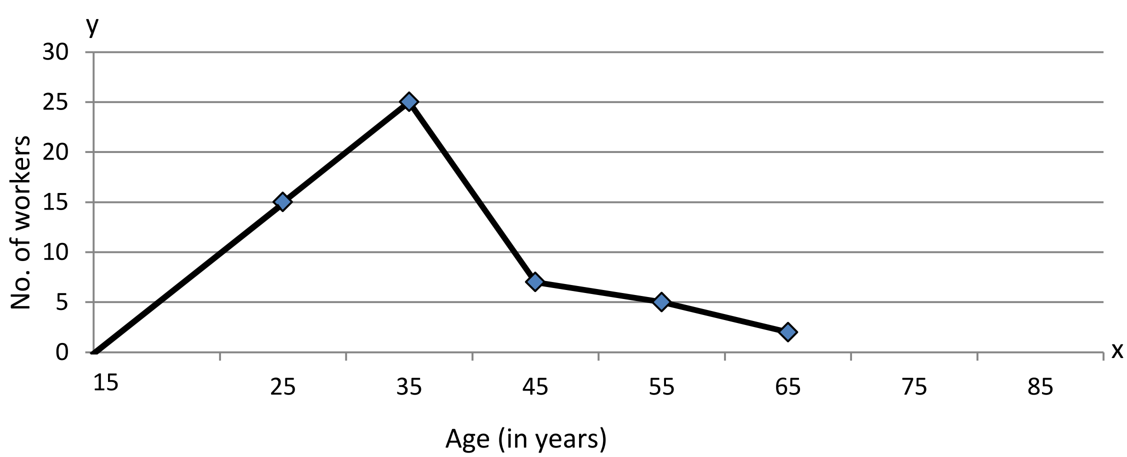
Step 3 : The figure so obtained is extended to the base at both ends by joining the extreme points (first and last point) to the mid-points of the two hypothetical classes (before the first class and after the last class), assuming that there frequencies are zero.

Ex : Construct a frequency polygon, without making the histogram, from the date given:-

AGE IN YEARS	20 – 30	30 – 40	40 – 50	50 – 60	60 – 70
NO. OF WORKERS	15	25	7	5	2

Solution : The mid-points of various class intervals (age) are taken along the x-axis and frequency (number of workers) corresponding to each mid-point is shown on the y-axis. The points, thus obtained are joined by straight lines to given a frequency polygon.

Frequency Polygon (without histogram) :



Frequency Curve: When the vertical of a frequency polygon are joined by a smooth curve, the resulting figure is known as a frequency curve.

a. Frequency curve may not necessarily pass through all the points of the frequency polygon but it passes through than as closely as possible.
b. The curve is made free hand in such a manner that the area included under the curve is approximately the same as that of the polygon.
c. This curve should start and end at the base line. As a general rule, it may be extended to the mid-points of class interval just outside the histogram.

d. The area under the curve should represent the total number of frequency in the whole distribution.

Frequency Polygon vs frequency curve: The only different between the two is that in case of frequency polygon, the points are joined by straight lines. However, in case of frequency curve, we make use of free hand to join these points.

Steps of Frequency Curve:

Step 1: Draw a Histogram with the given data.

Step 2: Obtain the mid-points of the upper horizontal side of each rectangle.

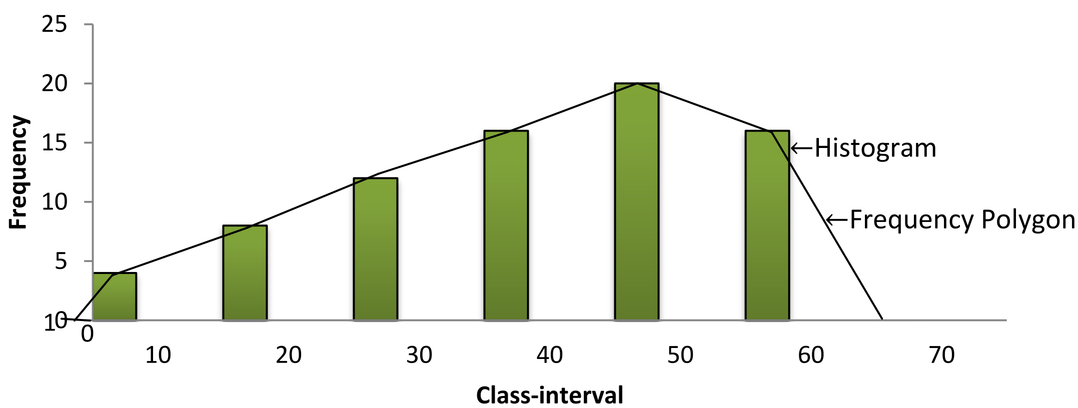
Step 3: Join these mid-points of the adjacent rectangle of the Histogram by smooth free hand to get frequency curve.

Step 4: Extend the ends of frequency curve the base line with the free hand.

Ex.: Draw a frequency curve from the following distribution:

CLASS INTERVAL	0 – 10	10 – 20	20 – 30	30 – 40	40 – 50	50 – 60
FREQUENCY	4	10	15	18	20	16

Solution: The frequency distribution is continues width equal class – intervals. We will first prepare a histogram and the frequency curve. The class intervals are shown along the x-axis and frequency on the y-axis.



Ogive or cumulative curve : The curve obtained by representing a cumulative frequency distribution on a graph is known as cumulative frequency curve or ogive.

Thus there are two types of cumulative frequency distribution viz, 'Less than' cumulative frequencies, we have accordingly two types of ogives:

(i) 'Less than' ogive
(ii) 'More than' ogive.

Less than ogive :

Step 1: In 'Less than' method frequencies of all the preceding class intervals are added to the frequency of a class.

Step 2: Plot 'Less than cumulated frequencies' against the upper limit of corresponding class interval.

Step 3:the points so obtained are joined by a smooth free hand curve to give 'Less than' ogive. 'Less than' ogive so obtained is an increasing curve, sloping upwards from left to right.

Ex: From the following distribution of monthly income of 60 people in a company. Draw a 'Less than' ogive curve.

MONTHLY INCOME (Rs. In '000'	No. OF PEOPLE
10 – 20	6
20 – 30	9
30 – 40	10
40 – 50	15
50 – 60	12
60 – 70	8

Solution: For depicting the 'Less than' ogive curve, above frequency distribution has to be converted into 'Less tha' cumulative frequency in the following manners.

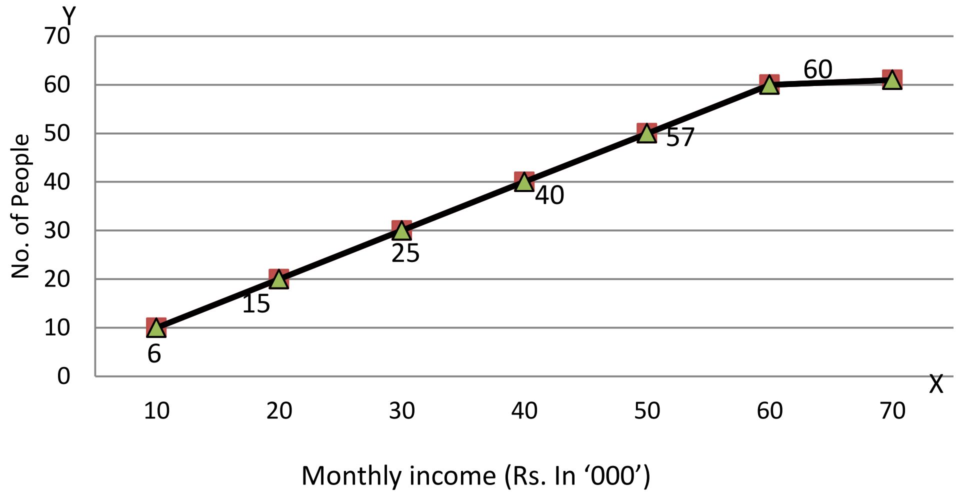
Cumulative Frequency Distribution

MONTHLY INCOME (Rs. IN '000')	NO. OF PEOPLE (c.f.)
Less than 20	6
Less than 30	15
Less than 40	25
Less than 50	40
Less than 60	52
Less than 70	60

‘Less than ‘ ogive showing monthly income of people

Scale: 1 cm. = Rs. 10,000 on x-axis.

1 cm. = 10 People on y-axis.



For constructing ‘Less than ogive’ , upper limit of the class interval is shown along the x-axis and the cumulative frequency of the respective class on the y-axis. The cumulative frequency will be plotted against the upper limit of the class-interval. Thus points are joined wide the help of free hand to get. ‘Less than, ogive.

More than ogive:

Step 1 : ‘In’ ‘more than ‘ method , frequencies of all the seceding class-interval are added to the frequency of a class i.e. , start with the lower limits of classes and go on subtracting frequencies of each classes.

Step 2 : Plot ‘More than cumulative frequencies’ against the lower limit of corresponding class interval.

Step 3 : The points so obtained are joined by a smooth free hand curve to give ‘More than ogive’. ‘More than’ ogive so obtained slept downwards from left to right.

Ex: Construct a 'More than' ogive from the below data:

Monthly Income (Rs. In '000'	No. of People
10 – 20	6
20 – 30	9
30 – 40	10
40 – 50	15
50 – 60	12
60 – 70	8

Solution : To construct the 'More than' ogive the frequency distribution will be converted into 'More than' cumulative frequency as shown in the following table:

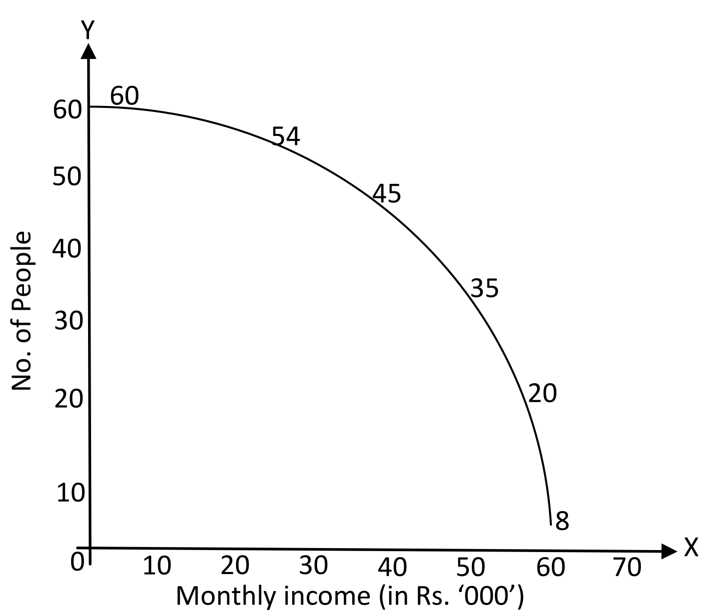
Cumulative Frequency Distribution

Monthly Income (Rs. In '000'	No. of People (c.f.)
More than 10	60
More than 20	54
More than 30	45
More than 40	35
More than 50	20
More than 60	8

'More Than Ogive showing monthly income of People

Scale : 1cm. = Rs. 10,000 on x-axis

1cm. = 10 People on y-axis.



CENTRAL BOARD OF SECONDARY EDUCATION

STUDY MATERIAL: APPLIED MATHEMATICS {Class XI}

UNIT 3: MEASURE OF CENTRAL TENDENCY

UNIT – 3

MEASURE OF CENTRAL TENDENCY

ArithmeticMean: Arithmetic Mean is defined as the sum of the value of all observations divided by the number of observations.

Calculation of Arithmetic mean in individual Series:

(i) **Direct Method:** $\bar{x} = \frac{\sum x}{N}$

Where $\sum x = x_1 + x_2 + \ldots\ldots + x_n$

N = Total number of items

Ex: The marks obtained by 10 students in a subject are :

STUDENTS	A	B	C	D	E	F	G	H	I	J
MARKS	85	60	50	75	55	40	53	70	45	65

Calculate Arithmetic mean by direct method

STUDENTS	MARKS (X)
A	85
B	60
C	50
D	75
E	55
F	40
G	55
H	70
I	45
J	65
N = 10	$\sum x = 600$

Arithmetic Mean $(\bar{x}) = \frac{\sum x}{N} = \frac{60}{10} = 60$ Marks

Short cut Method: Under this method, any figure is assumed as the mean and deviations are calculated from this assumed mean.

$$(\bar{x}) = A + \frac{\sum d}{N}$$

Where $(\bar{x})$ = Arithmetic mean, A = Assumed Mean d = X – A, deviation of variables from assumed mean, $\sum d = \sum(x - A)$, sum of the deviation from assumed mean; N = Total number of items.

<u>Ex</u>: Calculate the arithmetic mean of the marks by the assumed mean method.

<u>Solution:</u>

STUDENTS	MARKS (X)	d = X – A (A=50)
A	85	+35
B	60	+10
C	50 (A)	0
D	75	+25
E	55	+5
F	40	-10
G	55	+5
H	70	+20
I	45	-5
J	65	+15
N = 10		$\sum \boldsymbol{d} = \mathbf{100}$

$\bar{x} = A + \frac{\sum d}{N} = 50 + \frac{100}{10} = 60$ marks

<u>Step deviation Method:</u> In this method, deviations from assumed mean are divided by a common factor (c) to get step deviations.

Apply the following formula

$\bar{x} = A + \frac{\sum d'}{N} \times C$

d = X – A, i.e. , deviation of variable from assumed mean, d' = $\frac{X-A}{C}$ i.e. step deviation, $\sum d'$ = sum of step deviations, C = common factor.

<u>Note:</u> 'Step deviation method' can be used only when deviation from assumed mean (d) are divisible by a common factor.

<u>Ex:</u> Calculate the arithmetic mean of the marks by the step deviation method.

Solution:

STUDENTS	MARKS (X)	$d = X - A, A = 50$	$d^l = \frac{X-A}{C}$, $C = 5$
A	85	+35	+7
B	60	+10	+2
C	50 (A)	0	0
D	75	+25	+5
E	55	+5	+1
F	40	-10	-2
G	55	+5	+1
H	70	+20	+4
I	45	-5	-1
J	65	+15	+3
N = 10			$\sum d' = 20$

$$\bar{x} = A + \frac{\sum d\prime}{N} \text{ x } C$$

$$= 50 + \frac{20}{10} \text{ X } 5$$

$$= 60 \text{ marks.}$$

Discrete Series: In case of discrete series (ungrouped frequency distribution), values of variable shows the repetitions, i.e., frequencies are given corresponding to different values of variable. N = Sum total of frequency = $\sum f$

Direct Method:

$$\bar{x} = \frac{\sum fx}{\sum f}$$

Where$\sum fx$ = sum of the product of variable with the respective frequencies, $\sum f$ = Total number of items.

Ex: From the following data of the marks obtained by 60 students of a class, calculate the average marks by the direct method.

MARKS	20	30	40	50	60	70
NO. OF STUDENTS	8	12	20	10	6	4

Solution:

MARKS (X)	NO. OF STUDENTS (f)	(fx)
20	8	160
30	12	360
40	20	800
50	10	500
60	6	360
70	4	280
	$\sum f = 60$	$\sum f x = 2{,}460$

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{2{,}460}{60} = 41 \text{ marks.}$$

Shortcut Method: $\bar{x} = A + \frac{\sum fd}{\sum f}$

A = Assumed Mean, d = X – A i.e. , deviation of variable from assumed mean.

$\sum fd$ = sum of the product of deviation (d) with the respective frequencies (f) :

$\sum f$ = Total number of items.

Ex: Calculation of Arithmetic Mean of the marks by the short cut method.

Solution:

MARKS	NO. OF STUDENTS(f)	d = X – A (A=40)	fd
20	8	-20	-160
30	12	-10	-120
40(A)	20	0	0
50	10	+10	+100
60	6	+20	+120
70	4	+30	+120
	$\sum f = 60$		$\sum fd = +60$

$$\bar{x} = A + \frac{\sum fd}{\sum f} = 40 + \frac{+60}{60} = 41 \text{ marks.}$$

Step Deviation Method: in this method, the values of the deviations (d) are divided by a common factor (c) to case the calculation process.

$$\bar{x} = A + \frac{\sum fd'}{\sum f} \text{ x } C$$

d’ $=$ Step deviations (deviations from assumed mean divided by common factor)

$\sum fd^{1}$= sum of the product of step deviations (d’) with the respective frequencies (f)

<u>Ex:</u> Calculate the arithmetic mean of the marks by the step deviation method.

<u>Solution:</u>

MARKS X	NO. OF STUDENTS f	d = X – A (A=40)	$d^{1} = \frac{X-A}{C}$ (C=10)	fd’
20	8	-20	-2	-16
30	12	-10	-1	-12
40(A)	20	0	0	0
50	10	+10	+1	+10
60	6	+20	+2	+12
70	4	+30	+3	+12
	$\sum f = 60$			$\sum fd' = +6$

$$\bar{x} = A + \frac{\sum fd^{1}}{\sum f} \text{ x } C = 40 + \frac{+6}{60} \text{ x } 10 = 41 \text{ marks.}$$

<u>Continuous Series:</u> In the case of continuous series (grouped frequency distribution), the value of a variable is grouped in various class-intervals along with their respective frequencies. In a continuous series, the mid-points of the various class-intervals are used to replace the class interval. Once it is done, there is no difference between a continuous series and a discrete series.

In the continuous series also, the following three methods are used to calculate arithmetic mean.

(i) Direct method (ii) Short cut Method (iii) Step deviation Method.

(i) <u>Direct Method:</u> The following table given the marks in English secured by 30 students of a class in their weekly test:

MARKS	0 – 5	5 – 10	10 – 15	15 – 20	20 – 25
NO. OF STUDENTS	2	8	6	10	4

Calculation of Average marks (Direct Method)

MARKS X	NO. OF STUDENTS f	MID- VALUE (m)	f m
0 – 5	2	2.5	5
5 – 10	8	7.5	60
10 – 15	6	12.5	75
15 – 20	10	17.5	175
20 – 25	4	22.5	90
	$\sum$ f = 30		**$\sum$ f m = 405**

$$\bar{x} = \frac{\sum fm}{\sum f} = \frac{405}{30} = 13.50 \text{ marks.}$$

Shortcut Method: $\bar{x} = A + \frac{\sum fd}{\sum f}$

MARKS X	NO. OF STUDENTS f	MID-VALUE 'm'	d = m – A A = 12.5	f d
0 – 5	2	2.5	-10	-20
5 – 10	8	7.5	-5	-40
10 – 15	6	12.5(A)	0	0
15 – 20	10	17.5	+5	+50
20 – 25	4	22.5	+10	+40
	$\sum$ f =30			**$\sum$ fd = +30**

$$\bar{x} = A + \frac{\sum fd}{\sum f} = 12.5 + \frac{+30}{30} = 13.50 \text{ marks.}$$

Step Deviation Method: When class-intervals for all the classes in a continuous series are of same magnitude (width), then shortcut method can be further simplification by the step deviation method.

$$\bar{x} = A + \frac{\sum fd'}{\sum f} \text{ x C}$$

Ex:

MARKS X	NO. OF STUDENTS f	MID VALUE m	d = m – A (A= 12.5)	$d' = \frac{m-A}{C}$ (C=5)	f d'
0 – 5	2	2.5	-10	-2	-4
5 – 10	8	7.5	-5	-1	-8
10 – 15	6	12.5(A)	0	0	0
15 – 20	10	17.5	+5	+1	+10
20 – 25	4	22.5	+10	+2	+8
	$\sum$ f =30				**$\sum$ fd' = 6**

Average Marks $\bar{x} = A + \frac{\sum fd'}{\sum f} \times C = 12.5 + \frac{+6}{30} \times 5$

$= 13.50$ marks

Median: Median may be defined as the middle value in the data set when its elements are arranged in a sequential order, that is, in either as ascending or descending order of magnitude.

Computation of Median:

(i) Individual Series: Median (Me) = Size of $\left(\frac{N+1}{25}\right)$th item.

Odd and Even Number of items:

(i) In case of odd numbering items

Median = Middle item of distribution.

(ii) In case of Even number of items:

Median = Average of two middle terms.

Ex: Marks of five students are 42, 40, 60, 55 and 50 calculate median.

Step 1: Marks arranged in ascending order 40, 42, 50, 55, 60.

Step 2 : Me = size of $\left[\frac{N+1}{2}\right]$th item

Me = size of $\left[\frac{5+1}{2}\right]$th item = size of 3rd item.

Median = 50.

Discrete Series: In a discrete series, the values of the variable are given along with their frequencies.

Ex: Calculate the median from the following data :

SIZE (X)	FREQUENCY (f)
3	2
4	1
5	3
6	7
7	4

Solution:

SIZE (X)	FREQUENCY (f)	CUMULATIVE FREQUENCY
3	2	2
4	1	3
5	3	6
6	7	13
7	4	17

$$N = \sum f = 17$$

$\text{Me} = \text{size of } \left[\frac{N+1}{2}\right] \text{th item } = \text{size } \left[\frac{17+1}{2}\right] \text{th item.}$

Since 9th item falls in the cumulative frequency

$\therefore \text{Me} = 6.$

Continuous Series : Steps to calculate Median :

Step 1 : Arrange the data in ascending or descending order.

Step 2 : Calculate the c.f.

Step 3 : Find the median item as :

$$\text{Me} = \text{size of } \left[\frac{N}{2}\right] \text{th item.}$$

Step 4 : Find out c.f. which is either equal to or just greater them this.

Step 5: find the class corresponding to c.f. equal to $\frac{N}{2}$ or just greater than this. This class is called median class.

Step 6: $\text{Me} = l_1 + \frac{\frac{N}{2} - \text{c.f.}}{f} \text{x i}$

Ex: find the median of the following data :

X	0 – 10	10 – 20	20 – 30	30 – 40	40 – 50
f	3	4	2	7	9

Solution:

X	f	c.f
0 – 10	3	3
10 – 20	4	7
20 – 30	2	9
30 – 40	7	16
40 – 50	9	25

$$N = \sum f = 25$$

Median $= \frac{N}{2} = \frac{25}{2} = 12.5^{th}$ item.

12.5^{th} item lies in the group 30 – 40

$l_1 = 30,\ \ c.f. = 9, f = 7, i = 10$

By applying formula:

$$Me = l_1 + \frac{\frac{N}{2} - c.f.}{f} \times i$$

$$= 30 + \frac{12.5-9}{7} \times 10$$

$$= 35$$

Median $= 35$

Mode: mode is the value occurring most frequently in a set of observation and around which other items of the set cluster most density.

Calculation of mode:

(i) **Individual Series:**

(A) **Mode by way of observation:**

Step 1: Arrange the data in ascending or descending order.

Step 2: The item which occurs most in the series is 'Mode'.

Ex: From the heights of 15 students, calculate the value of mode.

Height (in inches) : 52, 50, 66, 70, 66, 72, 71, 66, 60, 67, 69, 67, 48, 60, 65.

Solution: By arranging the series in an ascending order we get.

48, 50, 52, 60, 60, 65, 66, 66, 66, 67, 67, 69, 70, 71, 72.

By observation, height 66 inches occurs most, therefore the mode (z) is 66 inches.

Mode by converting individual series into Discrete series :

Ex: Calculate the value of mode from the data by converting the data into discrete series:

Solution:

Heights (In inches)	Frequency
48	1
50	1
52	1
60	2
65	1
66	3
67	2
69	1
70	1
71	1
72	1

This height of 66 inches has the maximum frequency therefore, mode height i.e., (z) is 66.

Mode = 66 inches.

Discrete Series: There are two methods to determine mode in a discrete series:

(i) Mode by observation.

(ii) Mode by grouping method.

(i) Mode by observation:

Ex : find out mode of the following series?

Daily wages (in Rs.)	100	110	120	130	140	150
No. of Persons	2	4	8	10	5	4

Solution: By inspection, we can see that 130 occurs most frequently in the series, Hence the mode is Rs. 130.

(ii) Steps of grouping method:

Ex: Calculation the value of mode from the data by grouping method:

Solution:

GROUPING TABLE

Wages in (Rs.)	No. of Persons (f)	In Two's		In Three's		
	Column I	Column II	Column III	Column IV	Column V	Column VI
100	2	2+4=6		2+4+8=14		
110	4		4+8=12		4+8+10=22	
120	8	8+10=18				8+10+5=23
130	10		10+5=15	10+5+4=19		
140	5	5+4=9				
150	4					

In analysis table, we enter the value having maximum frequencies in each column of grouping table by means of (√).

Column No.	100	110	120	130	140	150
I				√		
II			√	√		
III				√	√	
IV				√	√	√
V		√	√	√		
VI			√	√	√	
Total		1	3	6	3	1

Since the value 130 has occurred the maximum number of time i.e. , 6, the modal income is Rs. 130.

Mode = Rs. 130.

Continuous Series : In continuous series, mode lies in a particular class or group, which is called the modal class. The following two methods are used in determining mode:

(i)**Observation Method:**

Step 1: Determine the modal class, i.e. , class with the highest frequency.

Step 2: Determine the exact value of mode by the following formula:

$$Mo = l_1 + \frac{f_1 - f_0}{2f_1 - f_0 - f_2} \times i$$

Where

Mo = Mode.

$\mathbf{l_1}$ = Lower limit of modal class.

$\mathbf{f_1}$ = Frequency of the modal class.

$\mathbf{f_0}$ = Frequency of class preceding the modal class.

$\mathbf{f_2}$ = Frequency of class succeeding the modal class.

i = Class-interval of the modal class.

Ex: find out mode of the following series :

Class-Interval	0 – 5	5 – 10	10 – 15	15 – 20	20 – 25
Frequency	2	4	15	6	7

Solution:

Class-Interval	**Frequency**
0 – 5	2
5 – 10	4 fo
10 – 15	15 f_1 Modal class
15 – 20	6 f_2
20 – 25	7

To calculate mode, the following formula will be used:

$$\text{Mode (z)} = l_1 + \frac{f_1 - f_0}{2f_1 - f_0 - f_2} \text{ x i}$$

$l_1 = 10, f_1 = 15, f_0 = 4, f_2 = 6, \ \ i = 5$

$$Z = 10 + \frac{15-4}{2 \text{ x } 15-4-6} \text{x } 5 = 10 + \frac{11}{20} \text{ x } 5$$

$$= 12.75$$

Mode = 12.75

Grouping Method:

Ex: From the following data, determine mode.

SIZE	10 – 20	20 – 30	30 – 40	40 – 50	50 – 60	60 –70	70 – 80	80 – 90
FREQUENCY	4	10	25	15	23	22	12	3

Solution:

GROUPING TABLE

SIZE X	FREQUENCY	IN TWO'S		IN THREE'S		
	Column I	Column II	Column III	Column IV	Column V	Column VI
10 – 20	4					
20 – 30	10	14	35			
30 – 40	25			39	50	63
40 – 50	15	40				
50 – 60	23		38			
60 – 70	22	45		60	57	37
70 – 80	12					
80 – 90	3	15	34			

Analysis Table

Column No.	10 – 20	20 – 30	30 – 40	40 – 50	50 – 60	60 – 70	70 – 80	80 – 90
I			√					
II					√	√		
III				√	√			
IV				√	√	√		
V					√	√	√	
VI			√	√	√			
Total			2	3	5	3	1	

It is clear that modal class is 50 – 60 and frequency of this class is 23

Using formula :

$$\text{Mode (z)} = l_1 + \frac{f_1 - f_0}{2f_1 - f_0 - f_2} \text{ x i}$$

$$l_1 = 50,\ f_1 = 23, f_0 = 15, f_2 = 22,\ i = 10$$

$$Z = 50 + \frac{23-15}{2 \text{ x } 23-15-22} \text{x } 10$$

$$= 50 + \frac{8}{9} \text{ X } 10 = 58.89$$

Mode = 58.89.

ANALYSIS OF DATA

Measure of Dispersion: The dispersion or scatter in a data is measured on the basis of the observations and the types of the measure of central tendency, used there. There are following measure of dispersion;The runs scored by two batsman in their last ten matches as follows:

Batsman A: 30, 91, 0, 64, 42, 80, 30, 5, 117, 71.

Batsman B: 5 3, 46, 48, 50, 53, 53, 58, 60, 57, 52.

(i) **Range:**The runs scored by two batsmen A and B, we had some idea of variability in the scored on the basis of minimum and maximum runs in each series. To obtain a single number for this, we find the difference of maximum and minimum value of each series. This difference is called the 'Range' of the data.

In case of batsman A Range = 117-0=117 and for batsman B Range =60 – 46 = 14 clearly, Range of A > Range of B. Therefore the scores are scattered or dispersed in case of A while for B these are close to each other.

Thus, Range of series = maximum value – minimum value the range of data gives us a rough idea of variability or scatter but does not tell about the dispersion of the data from a measure of central tendency. For this purpose, we need some other measure of variability clearly, such measure must depend upon the differences (or deviation) of the value from the central tendency.

Mean Deviation: The deviation of an observation x from a fixed value 'a' is the difference x –a. remember that in finding a suitable measure of dispersion, we require the distance of each value from a central tendency or a fixed number 'a' Recall that the absolute value of the difference of two numbers gives the distance between the numbers when represented on a number line. Thus to find the measure of dispersion from a fixed number 'a' we may take the mean of the absolute values of the deviations from the central values. This mean is called the 'mean deviation'. The mean deviation from 'a' is denoted as M.D. (a). Therefore

$$\textbf{M.D. (a)} = \frac{\textbf{sum of absolute value of deviation from 'a'}}{\textbf{Number of observations}}$$

Mean deviation may be obtained from any measure of central tendency. However, mean deviation from mean and median are commonly used in statistical studies.

Mean Deviation for ungroupteddata: Let n observation be x^1, x^2........,xn. The following steps are involved in the calculation of mean deviation about mean or median.

Step 1: Calculate the measure of central tendency about which we are to find the mean deviation let it be 'a'.

Step 2: Find the deviation of each xi from a, i.e. x_1 -a, x_2 – a,....,x_n - a.

Step 3: Find the absolute values of the deviations, i.e., drop the minus sign (-), if it is there, i.e., |x_1-a |,| x_2 – a | ,....., | xn-a |.

Step 4: find the mean of the absolute value of the deviations. This mean is the mean deviation about a, i.e.,

$$\text{M.D. (a)} = \frac{\sum_{i=1}^{n} |x_i - a|}{n}$$

Thus M.D. ($\bar{x}$) = $\frac{1}{n}\sum_{i=1}^{n} | x_i - \bar{x} |$, where $\bar{x}$ = means and

M.D. (M) = $\frac{1}{n}\sum_{i=1}^{n} | x_i - M|$, where M = Median.

Ex1= find the mean deviation about the mean for the following data: 6,7,10,12,13,4,8,12.

Solution: We precedestep wise and get the following:

Step1: mean of the given data is

$$\bar{x} = \frac{6+7+10+12+13+4+8+12}{8} = \frac{72}{8} = 9$$

Step 2: The deviation of the respective observation from the mean$\bar{x}$, i.e. x_i - $\bar{x}$ are 6-9, 7-9, 10-9, 12-9, 13-9, 4-9, 8-9, 12-9 or -3, -2, 1, 3, 4, -5, -1, 3.

Step 3 : The absolute value of the deviations i.e. $|x_i - \bar{x}|$ are 3, 2, 1, 3, 4, 5, 1, 3.

Step 4: The required mean deviation about the mean is

$$\text{M.D.}(\bar{x}) = \frac{\sum_{i=1}^{8} |x_i - \bar{x}|}{8}$$

$$= \frac{3+2+1+3+4+5+1+3}{8}$$

$$= \frac{22}{8}$$

$$=2.75$$

Ex2 : find the mean deviation about the median for the following data : 3, 9, 5, 3, 12, 10, 18, 4, 7, 19, 21.

Solution: Here the number of observation is || which is odd. Arranging the data into ascending order , we have 3, 3, 4, 5, 7, 9, 10, 12, 18, 19, 21

Now Median = $\left(\frac{11+1}{2}\right)$th or 6th observation = 9.

The absolute value of the respective deviation from the median, i.e. , $|x_i$ - M$|$ are 6, 6, 5, 4, 2, 0, 1, 3, 9, 10 , 12.

Therefore $\sum_{i=1}^{11} |x_i - M| = 58$

And M.D. (M) = $\frac{1}{11}\sum_{i=1}^{11} |x_i - M| = \frac{1}{11}\sqrt{58} = 5.27$.

Mean deviation for grouped data:we know that data can be grouped into two ways:

(a) Discrete frequency distribution.
(b) Continuous frequency distribution.

(a) Discrete frequency distribution : Let the given data consist of n discrete values x_1, x_2….., xn occurring with frequencies f_i, f_2……., f_n respectively. This data can be represented in the tabular form as given below, and is called discrete frequency distribution.

$\underline{x}: x_i x_2 x_3 \ldots\ldots x_n$

$\int : f_i, f_2, f_3 \ldots\ldots f_n$

(i) **Mean deviation about mean** : first of all we find the mean $\bar{x}$ of the given data by using the formula .

$$\bar{x} = \frac{\sum_{i=1}^{n} xi\, f_i}{\sum_{i=1}^{n} fi} = \frac{I}{N}\sum_{i=1}^{n} xi\, f_i$$

Thus, we find the deviation of observation x_i from the mean $\bar{x}$ and take their absolute values i.e. $|x_i - \bar{x}|$ for all i = 1,2,3……,n.

After this find the mean of the absolute value of the deviation, which is the required mean deviation about the mean , Thus

$$\text{M.D. }(\bar{x}) = \frac{\sum_{i=1}^{n} f\text{i}\, |x_i - \bar{x}|}{\sum_{i=1}^{n} f\text{i}} = \frac{i}{N}\sum_{i=1}^{n} f\text{i}\, |x_i - \bar{x}|$$

(ii) **Mean deviation about median**: To find mean deviation about median, we find the median of the given discrete frequency distribution. For this the observation are arranged in ascending order. After this the cumulative frequency are obtained. Thus we identify the observation whose cumulative frequency is equal to or just greater than $\frac{N}{2}$, where N is the sum of the frequencies. This value of the observation lies in the middle of the data.

Therefore, it is the required median, After finding median, we obtain the mean of the absolute value of the deviation from median, Thus ,

$$\text{M.D. (M)} = \frac{I}{N}\sum_{i=1}^{n} f\text{i}\, |x_i - \text{M}|$$

Ex1 : Find mean deviation about the mean for the following data:

x_i :	2	5	6	8	10	12
f_i :	2	8	10	7	8	5

Solution:

x_i	f_i	$f_i x_i$	$\|x_i - \bar{x}\|$	$f_i \|x_i - \bar{x}\|$
2	2	4	5.5	11
5	8	40	2.5	20
6	10	60	1.5	15
8	7	56	0.5	3.5
10	8	80	2.5	20
12	5	60	4.5	22.5
	40	**300**		**92**

$N = \sum_{i=1}^{6} f\ i = 40$, $\sum_{i=1}^{6} f\text{i}x_i = 300$, $\sum_{i=1}^{6} f\text{i}\ |x_i - \bar{x}\ | = 92$

Therefore $\bar{x} = \frac{I}{N}\sum_{i=1}^{6} f\text{i}x_i = \frac{1}{40}$ x 300= 7.5

And M.D. $|\bar{x}\ | = \frac{I}{N}\sum_{i=1}^{6} f\text{i}\ |x_i - \bar{x}\ | = \frac{1}{40}$ x 92 = 2.3

<u>Ex:</u> Find the mean deviation about the median for the following data:

x_i :	3	6	9	12	13	15	21	22
f_i:	3	4	5	2	4	5	4	3

Solution: The given observation are already in ascending order. We get cumulative frequencies of the data by adding a row:

x_i	3	6	9	12	13	15	21	22
f_i	3	4	5	2	4	5	4	3
c.f	3	7	12	14	18	23	27	30

Now, N =30 which is even median is the mean of the 15th and 16th observations. Both of these observations lie in the cumulative frequency 18, for which corresponding observation is 13.

Therefore, median M = $\frac{15^{th}\text{ observation} + 16^{th}\text{ observation}}{2}$

$= \frac{13 + 13}{2}$

$= 13$

Now, absolute values of the deviation from median, i.e. $|x_i$ - M$|$ are given below:

$\|x_i$ - M$\|$	10	7	4	1	0	2	8	9
$\int i$	3	4	5	2	4	5	4	3
$\int i\|x_i$ - M$\|$	30	28	20	2	0	10	32	27

We have $\sum_{i=1}^{8} f\text{i} = 30$ and $\sum_{i=1}^{8} f\text{i}\ |x_i - \text{M}| = 149$

Therefore M.D. (M) $= \frac{I}{N}\sum_{i=1}^{8} f\text{i}\ |x_i - \text{M}|$

$= \frac{1}{30}$ x 149 = 4.97

(b) **<u>Continues frequency distribution</u>**: A continues frequency distribution is a series in which the data are classified into different class intervals without gaps along with their respective frequencies.

(i) **Mean deviation about mean**: While calculating the mean of a continuous frequency distribution we had made the assumption that the frequency in each class is centered at its mid – point. Here also we write the mid-point of each given class and proceed further as for a discrete frequency distribution to find the mean deviation.

Ex: find the mean deviation about the mean for the following data.

Marks obtained : 10-20 20-30 30-40 40-50 50-60 60-70 70-80.
Number of students: 2 3 8 14 8 3 2.

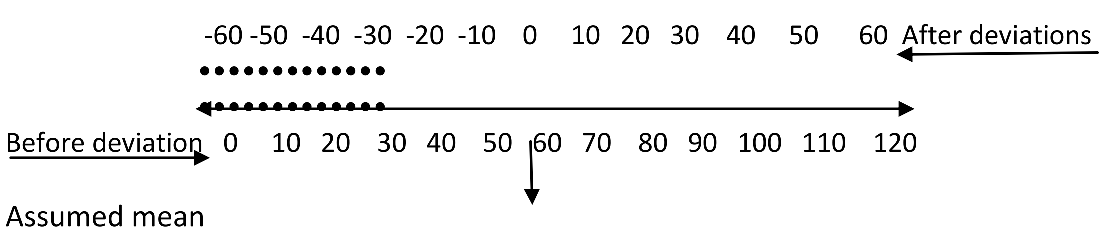
MARKS OBTAINED	NUMBER OF STUDENTS f_i	MID – POINT x_i	$f_i x_i$	$\lvert x_i - \bar{x} \rvert$	$f_i \lvert x_i - \bar{x} \rvert$
10-20	2	15	30	30	60
20-30	3	25	75	20	60
30-40	8	35	280	10	80
40-50	14	45	630	0	0
50-60	8	55	440	10	80
60-70	3	65	195	20	60
70-80	2	75	150	30	60
	40		**1800**		**400**

Here N = $\sum_{i=1}^{7} f$i = 40, $\sum_{i=1}^{7} f$i x_i= 1800, $\sum_{i=1}^{7} f$i $\lvert x_i - \bar{x} \rvert$= 400

Therefore $\bar{x} = \frac{I}{N}\sum_{i=1}^{7} f$i x_i = $\frac{1800}{40}$ = 45

And M.D. ($\bar{x}$) = $\frac{I}{N}\sum_{i=1}^{7} f$i $\lvert x_i - \bar{x} \rvert$ = $\frac{1}{40}$ x400 = 10

Short cut method for calculating mean deviation about mean: we can avoid the tedious calculation of computing $\bar{x}$ by following step deviation method. In this method, we take an assumed mean which is in the middle or just class to it in the data. Then deviation of the observation (or mid-points of classes) are taken from the assumed mean. This is nothing but the shifting of origin from zero to the assumed mean on the number line.



If there is a common factor of all the deviation, we divide them by this common factor to further simplify the deviations. These are known as step deviations. The process of taking step deviation is the change of scale on the number line as shown below.

Deviation from -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 Step deviation

Assumed mean -60 -50 -40 -30 -20 -10 0 10 20 30 40 50 60

0 10 20 30 40 50 60 70 80 90 100 110 120

Assumed Mean

The deviation and step –deviation reduce the size of the observation, so that the computation became simpler. Let the new variable be denoted by di = $\frac{xi-a}{h}$, where 'a' is the assumed mean and h is the common factor. Thus, the mean $\bar{x}$ by step deviation method is given by

$$\bar{x} = a + \frac{\sum_{i=1}^{n} f\text{i}\, d_i \times h}{N}$$

Ex: find the mean deviation about mean for the following data by using sep deviation method.

Marks obtained:	10 –20	20-30	30-40	40-50	50-60	60-70	70-80
Number of students:	2	3	8	14	8	3	2

Solution:

MARKS OBTAINED	NUMBER OF STUDENTS	MID–POINT	$d_i = \frac{x-45}{10}$	$f_i d_i$	$\lvert x_i - \bar{x} \rvert$	$f_i \lvert x_i - \bar{x} \rvert$
10 – 20	2	15	-3	-6	30	60
20 – 30	3	25	-2	-6	20	60
30 – 40	8	35	-1	-8	10	80
40 – 50	14	45	0	0	0	0
50 – 60	8	55	1	8	10	80
60 – 70	3	65	2	6	20	60
70 – 80	2	75	3	6	30	60
	40			**0**		**400**

Taking the assumed mean a = 45 and h = 10

Therefore $\bar{x} = a + \frac{\sum_{i=1}^{7} f\text{i}\, d_i \times h}{N}$

$= 45 + \frac{0}{40} \times 10 = 45$

And M.D. $(\bar{x}) = \sum_{i=1}^{7} f\text{i}\, \lvert x_i - \bar{x} \rvert = \frac{400}{40} = 10$

Mean deviation about median : The data is first arranged in ascending order. Then, the median of continuous frequency distribution is obtained by first identify the class in which median lies (median class) and then applying the formula.

Median= l + $\frac{\frac{N}{2}-c}{f}$x h

Where median class is the class internal whose cumulative frequency is just greater than or equalto $\frac{N}{2}$, N is the sum of frequencies, l, f, h and c are respectively lower limit the frequency, the width of the median class and c the cumulative frequency of the class just preceding the median class. After finding the median the absolute value of the deviation of mid-point x_i of each class from the median i.e. |x_i - M| are obtained.

Thus M.D. (M) = $\frac{I}{N}\sum_{i=1}^{n} f$i |$x_i$ - M|

Ex : Calculate the mean deviation about median for the following data:

Class :	0 – 10	10 – 20	20 – 30	30 – 40	40 – 50	50 – 60
Frequency:	6	7	15	16	4	2

Solution:

CLASS	FREQUENCY f_i	CUMULATIVE FREQUENCY c.f.	MID – POINT x_i	|x_i- Med.|	f_i|x_i - Med|
0 – 10	6	6	5	23	138
10 – 20	7	13	15	13	91
20 – 30	15	28	25	3	45
30 – 40	16	44	35	7	112
40 – 50	4	48	45	17	68
50 – 60	2	50	55	27	54
	50				**508**

The class interval containing $\frac{N}{2}$th or 25th item is 20 – 30. Therefore, 20 – 30 is the median class we know that

Median = l + $\frac{\frac{N}{2}-c}{f}$x h

Here l = 20, c= 13, f=15, h=10, and N = 50

Therefore Median = 20 + $\frac{25-13}{15}$ x 10 = 20 + 8 = 28

Thus , mean deviation about median is given by:

M.D. (M) = $\frac{1}{N}\sum_{i=1}^{6} f$i $|x_i$ - M$|$ = $\frac{1}{50}$ x 508 = 10.16

Limitation of mean deviation: In a series where the degree of variability is very high, the median is not a representative central tendency. Thus, the mean deviation about median calculated for such series can not be fully relied . Mean deviation is calculated on the basis of absolute value of the deviations and therefore, cannot be subjected to further algebraic treatment. This implies that we muet have some other measure of dispersion. Standard deviation is such a measure of dispersion.

Skewness :Skewness or asymmetry denoted the tendency of a distribution to depart from symmetry.

A frequency distribution is said to be skewed if the frequencies decrease with markedly greater rapidity on one side of the central maximum than on the other side. This characteristic of a frequency distribution is known as skewness and the measure of asymmetry are usually called 'measure of skewness'.

The main object of measuring skewness is to know the direction of the variation from an average and to compare the frequency distribution and shape of their curves.

Types of Skewness :

(i) **Positive Skewness** : Where in a frequency distribution the frequencies are piled up at the lower value of the variable and spread out over a greater range of values on the high value and the skewness is known as positive.

(ii) **Negative Skewness** : Where in frequency distribution the frequencies are piled up at the higher values of the variable and spread out over a greater range of value on the low value and the skewness is known as negative. That is in negatively skewed frequency distribution, the skewness is said to be negative.

Measure of sknewness :Measure of skewness may be subjective (or obstract or absolute) or relative. Absolute measure tell us the direction and extend of asymmetry in a frequency distribution. Relative measure often called coefficient of skewness, permit us to compare two or more frequency distributions.

First Measure of skewness (Karl Pearson's coefficient of skewness) :

(a) Absolute Measure, Sk = Mean – Mode

$= \bar{x} - z$

Relative Measure , J = $\frac{\text{Mean}-\text{Mode}}{\text{Standard Deviation}}$ or $\frac{\bar{x} - z}{\sigma}$

(b) If mode is not determined, then

Absolute measure ,Sk = 3(Mean – Median)

or

$= 3\ (\bar{x} - m)$

Relative Measures, J = $\frac{3(\text{Mean}-\text{Median})}{\text{Standard Deviation}}$ or $\frac{3\ (\bar{x} - m)}{\sigma}$

Ex: From the following data calculate Karl Pearson's coefficients of skewness:

YEAR	1950	1951	1952	1953	1954	1955	1956	1957	1958	1959
PRICE INDEX OF WHEAT	83	87	93	104	106	109	118	124	126	130

Solution :

S.NO.	**X**	**$(X - \bar{X})$**	**$(X - \bar{X})^2$**
1	83	-25	625
2	87	-21	441
3	93	-15	225
4	104	-4	16
5	106	-2	4
6	109	1	1
7	118	10	100
8	124	16	256
9	126	18	324
10	130	22	484

arithmetic Mean . $\bar{X} = \frac{\sum x}{N} = \frac{1080}{10} = 108$

Median, M $= \frac{N+1}{2}$th value

$= \frac{10+1}{2}$th value $= 5.5^{th}$ value

$$M = \frac{\text{5th value} + \text{6th value}}{2}$$

$$= \frac{106+104}{2} = \frac{215}{2} = 107.5$$

$$\text{S.D.}, \sigma = \frac{\sqrt{\sum(x - \bar{x})^2}}{N} = \frac{\sqrt{2476}}{10}$$

$$= \sqrt{247.6}$$

$$= 15.73$$

$$J = \frac{3\ (\text{Mean} - \text{Median})}{\text{Standard Deviation}}$$

$$= \frac{3\ (108 - 107.5)}{15.73}$$

$$= \frac{3 \text{ X } 0.5}{15.73}$$

$$= 0.0954$$

CENTRAL BOARD OF SECONDARY EDUCATION

STUDY MATERIAL: APPLIED MATHEMATICS {Class XI}

UNIT 4: COMMERCIAL MATHEMATICS

UNIT – 4

COMMERCIAL MATHEMATICS

Profit and Loss: In our everyday life and in the business world, we encounter transactions involving sales and purchases. Every time such a transaction occurs, it may be observed that there is a seller and a buyer involved.

The seller sells some things goods for a certain amount paid by the buyer. The seller eventually makes some profit or loss in the transaction.

Cost Price (CP): The price at which an article in purchased is called its cost price.

Selling Price (SP): The price at which an article is sold is called its Selling Price.

Profit: If the selling price of an article is greater than its cost price, we say that there is profit or gain.

Profit = Selling price – Cost price

Percentage of profit is always calculated on the cost price of an article.

Where SP>CP

1. Profit = SP – CP.
2. Profit percentage = $\frac{Profit}{CP}$ (100)%.
3. Profit = Profit percentage (CP)
4. When CP and profit percentage are given

   SP= (CP) $\left(\frac{100+profit\ percentage}{100}\right)$
5. When SP and profit percentage are given

   CP = $\frac{100\ (SP)}{100+profit\ percentage}$

Loss: If the selling price of an article is less than cost price, we say that there is a loss.

Loss = Cost price – Selling price

Percentage of loss is always calculated on the cost price of the article.

When SP<CP

1. Loss = CP -SP

2. Loss percentage $= \frac{Loss}{CP}$ x 100%
3. Loss = Loss percentage x CP
4. When CP and loss percentage are given,

   SP = CP$\left(\frac{100-Loss\ percntage}{100}\right)$
5. When SP and Loss percentage are given,

   CP =$\left(\frac{SP\ (100)}{(100-Loss\ percentage}\right)$

Overheads: All the expenditure incurred on transportation, repairs, etc., (if any) are categorized as overheads. These overheads are always included in the CP of the article.

Note: when there are two articles having the same cost price and, if one article is sold at a% profit and the other is sold at the same loss percent, then effectively neither profit nor loss is made. If there are two article having the same selling price and one is sold at x% profit and the other is sold at x% loss effectively, there is always a loss and the loss percentage is$\left(\frac{x}{10}\right)^2$%.

Ex: Rakesh purchased a TV for Rs. 5000,it paid Rs. 250 its transportation if he sold the TV for Rs. 5075, then find his profit or loss percentage.

Solution: Price at which the TV was bought = Rs. 5000

Overheads in the form of transportation = Rs. 250.

∴ The total cost price of the TV = (5000 + 250) = Rs. 5250

Selling price of the TV = Rs. 5075

SP<CP⟹ there is a loss.

The amount of loss = CP – SP

= 5250 – 5075

= Rs. 175

∴ Loss percentage = $\frac{Loss}{CP}$ x (100)%

= $\frac{175}{5250}$ (100)%

= $\frac{10}{3}$%

= 3.33%

∴ Rakesh incurred a loss of 3.33%.

Ex: By selling 24 pens, kranthi's lost an amount equal to the CP of 3 Pens. Find his loss percentage?

Solution: let us assume that cost price of each pen is Rs. 1.

$\Longrightarrow$ CP of 24 pens = Rs. 24

Loss = CP of 3 pens = 3 x 1 = Rs. 3

$\Longrightarrow$ Loss percentage = $\left(\frac{Loss}{CP} \text{ x } 100\right)\%$

$= \frac{3}{24} \text{ x } 100\%$

= 12.5%

∴ Kranthi's Loss is 12.5%.

Ex: Naresh sold two books for Rs. 600 each, thereby gaining 20% on one book and losing 20% on the other book. Find his overall loss or gain percent.

Solution: Selling price of the first book = Rs. 600 profit = 20%,

$\Rightarrow$ CP = $\frac{100\ SP}{100+Profit\ Percetage)}$

$$\Rightarrow \frac{100\ x\ 600}{100+20}$$

$\Rightarrow$Rs. 500.

The selling price of second book = Rs. 600, Loss – 20%

$\Longrightarrow$ CP = $\frac{100\ x\ SP}{(100-Loss\ percentage)} = \frac{100\ x\ 600}{100-20}$ = Rs. 750

So, the total cost price of the books = Rs. 500 + Rs. 750

= Rs. 1250

The total selling price of the books = 2 x 600 = Rs. 1200

As the total selling price of the books < the total cost price of the books, there is a loss.

Loss = CP – SP

= Rs. 1250 – Rs. 1200

= Rs. 50.

$\Longrightarrow$ Loss percentage = $\frac{Loss}{CP}$ (100)%

= $\left(\frac{50}{1250}\right)$100%

= 4%

$\therefore$ Naresh's loss is 4%.

Simple interest and compound interest:

Interest: Interest is paid to the lender by the borrower for using money for a specified period of time. Various terms and general representation are as follows:

1. Principal: The original sum borrows and denoted by P.
2. Time: The time for which money is borrowed. It is denoted by n.
3. Rate: The rate at which interest is calculated on the original sum. It is denoted by r.
4. Amount: The sum of principal and interest. It is denoted by A.

Simple Interest: When interest is calculated every year on the original principal, such interest is called Simple Interest. Here year after year, earn through the interest gets accumulated and is due to the lender, this accumulated interest is not taken into account for the purpose of calculating interest for the later years.

Simple Interest = $\frac{PNR}{100}$

Total Amount A = P + $\frac{PNR}{100}$

= P $\left(1+\frac{NR}{100}\right)$

Compound Interest : In compound interest, the interest is added to the principal at the end of each period to arrive at the new principal for the next period.

In other words, the amount at the end of the first year will become the principal for the second year; the amount at the end of the second year becomes then principal for the third year, and so on.

If P denoted the principal at the beginning of period 1, then

P at the beginning of year 2 = P $(1 + \frac{r}{100}) = PR$

P at the beginning of year 3 = P $(1 + \frac{r}{100})^2 = PR^2$

P at the beginning of year (n + 1) = P $(1 + \frac{r}{100})^n = PR^n$

When R = $\left\{ 1 + (\frac{r}{100}) \right\}$

Hence, the amount after n year = PR^n = A

Interest = I = A – P = P $(R^n - 1)$

Ex : If Rs. 5000 become Rs. 5700 in a year's time at simple interest, what will Rs. 700 become at the end of 5 years at the same rates of interest?

Solution: Amount = Principal + Interest

Principal = Rs. 5000

Therefore, 5700 = 5000 + Interest.

Interest = Rs. 700

$I = \frac{PNR}{100} \Longrightarrow 700 = \frac{5000 \, x \, R \, x \, 1}{100}$

$\Longrightarrow$ R = 14%.

∴ The simple interest on Rs. 7000 at the rate of 14% for a period of 5 years is

$\frac{7000 \, x \, 14 \, x \, 5}{100}$ = Rs. 4900. Therefore Rs. 7000

At the end of 5 years, amount to Rs. 7000 + 4900 = Rs. 11,900.

Ex: The cost of an electronic device reduces at the rate of 5% per annum. If it's present cost is Rs. 32,490. What was its cost before two years?

Solution:

$32{,}490 = P \left\{ 1 - \frac{R}{100} \right\}^2$

$32{,}490 = P \left\{ 1 - \frac{5}{100} \right\}^2$

$\Longrightarrow \quad 32{,}490 = P \left(\frac{20-1}{20}\right)^2$

$\Longrightarrow \quad 32{,}490 = P \times \frac{19}{20} \times \frac{19}{20}$

$\therefore$ P = Rs. 36,000.

In this formula $A = P\left(1 + \frac{r}{100}\right)^n$, the rate (r) will not be for one year but for the time period over which compounding is done and the power to which the term inside the bracket is raised will not be the number of years, but the number of years multiplied by the number of times compounding is done per year.

When compounding is done more than once a year, the rate of interest given in the problem is called nominal rate of interest.

We can also calculate the rate of interest which will yield simple interest in one year equal to the interest obtained under the compound interest at the given nominal rate of interest. The rate of interest so calculated is called effective rate of interest.

Present value, Future value : If we deposit a sum of money with the present value PV in a bank that pays interest at the rate r, then after one year it will become PV (1+r). Let us call this amount its future value FV. We may write it as:

$$FV = PV\,(1 + r)$$

We may also think of (1+r) as a growth factor. Continuing this process for another year, compounding the interest annually, the future value will become:

$$FV = [PV\,(1+r)\,(1+r)] = PV\,(1+r)^2$$

This gives the future value after two years. It we can continue this compounding for n years, the future value then becomes:

$$FV = PV\,(1+r)^n \ldots\ldots\ldots(i)$$

The above expression is valid for annual compounding. If we do the compounding quarterly, the amount of interest credited will be only at the rate r/4, but there will also be 4n compounding periods in n years. Similarly, for monthly compounding the interest rate is $\frac{r}{12}$ per month and the compounding occurs 12n times in n years. Thus, the above equation becomes:

$$FV = PV\left(1+\frac{r}{12}\right)^{12n}.$$

From eqn. (i) , we can get

$$PV = \frac{FV}{(1+r)^n} \ldots\ldots\ldots\ldots(ii)$$

This gives the present value of a future payment. Discounting is the procedure to convert. The future value of a sum of money to its present value.

Discounting is a very important concept in finance because it allows us to compare the present value of different future payments.

Annuity: In many financial situations, we have to deal with a stream of payments, such as rent receipts, or monthly pay checks. An annuity represents such a series of cash payment, even for monthly or weekly payments. Another example of an annuity is that of a loan that you take out and then pay back in monthly installments. Many insurance companies give the proceeds of a lift insurance policy either as a lump sum, or in the form of an annuity.

If there is a cash flow C at the end of first, second, third…. Period, thus the sum of discounted cash flows is given by:

$$S = \frac{C}{1+r} + \frac{C}{(1+r)^2} \rightarrow \text{...... + n terms (i)}$$

Here S represents the present value of all future cash flows. We compare it to the standard form of geometric series.

$$S = a + ax + \text{........} + ax^{n-1} \text{ .... (ii)}$$

We notice that the first term a = $\frac{C}{1+r}$, and the ratio between the terms x = $\frac{1}{1+r}$, we know its summation as

$$Sn = \frac{a\,(1-x^n)}{1-x} \text{ .... (iii)}$$

This gives $$S = \frac{\frac{C}{1+r}\,(1-\frac{C}{(1+r)^n})}{1-\frac{1}{1+r}}$$

$$\Longrightarrow S = \frac{C\,[1-(1+r)^{-n}]}{r} \text{................(iv)}$$

Using the sigma rotation for summation, we may write (i) as

$$S = \frac{C}{1+r} + \frac{C}{(1+r)^2} \rightarrow \text{........... n terms}$$

$$= \sum_{i=0}^{n} \frac{C}{(1+r)^{j,1'}}$$

Thus we get

$\sum_{i=1}^{n} \frac{C}{(1+r)^j} = \frac{C\,[1-(1+r)^{-n}]}{r}$

We can use the formula (iii) to find the future value of an annuity.

<u>Ex</u>: **<u>Present value, Interest rate</u>**: You expect to receive Rs. 10,000 as a bonus after 5 years on the job. You have calculated the present value of this bonus and the answer is Rs. 8000. What discount rate did you use in your calculation?

To find the PV of a future sum of money, we use

$PV = \frac{FV}{(1+r)^n}$

This gives $8000 = \frac{10{,}000}{(1+r)^5}$

Or $(1+r)^5 = \frac{10{,}000}{8000} = 1.25$

$1 + r = (1.25)^{1/5} = 1.0456$

And thus r = 4.56%

<u>Depreciation</u>: Depreciation refers to the aspects of the same concept, first, the actual decrease in value of fair value of an asset, such as the decrease in value of factory equipment each year as it is used and wears, and second the allocation in according statement of the original cost of the assets to periods in which the assets are used.

Depreciation is thus the decrease in the value of assets and the method used to reallocation the cost of a tangible asset over its useful life span. Generally the cost is allocated as depreciation expense among the periods in which the asset is expected to be used. Depreciation has been defined as the diminution in the utility or value of an asset, and is a non-cash expense.

<u>Method of providing or Allocating Depreciation</u> – Different method are suitable for different assets depending upon the nature and type of the asset.

Straight Line Method – This method is also termed as original cost method' because under this method depreciation is changed at a fixed percentage on the original cost of the asset. The amount of depreciation remains equal from year to year and as such the method is also known as Equal installment method or 'fixed installment method '. Under this method, the amount of depreciation is calculated by deducting the scrap values from the original cost of the asset and then by dividing the remaining balance by the number of years of its estimated life. The depreciation so calculate and changed annually will rules the original cost of the asset to zero , or its scrap value , as the case may be at end of its estimated life.

Under this method, the amount of depreciation is calculated by the following formula:

Year Depreciation = $\frac{\text{original cost of the asset} - \text{Estimated scrap value}}{\text{Estimated life of the asset.}}$

for example , if the original cost of the asset is Rs. 100,000 and its scrap value is likely to be Rs. 15,000 after its estimated life of 10 years , the depreciation written off will be $\frac{10000-15000}{10}$= Rs. 8500 every years.

Ex: On 1st April 2015,Atul Glass Limited purchased a machine for Rs. 90,000 and spent Rs. 6000 on its carriage and Rs. 4000 on its erection. On the date of purchase , it was estimate that the effective life of the machine will 10 years and after 10 years its scrap value will be Rs. 20000/-.

Prepare machine A/C and depreciation A/C for 4 years after providing depreciation on fixed installment method.

Accounts are closed on 31st march every year.

Solution: As the rate of depreciation is not given in the question, the amount of annual depreciation will be arrived at as under:

Annual Depreciation = $\frac{\text{Cost of asset} - \text{Scrap value}}{\text{Estimated life of Asset}}$

$= \frac{8000}{100000} \text{x } 100$

= 8 %

Special features or characteristics of depreciation:

1. Depreciation is decline in the value of fixed assets (except land).
2. Such fall is of a permanent nature. Once the value of an asset is reduced due to depreciation, it cannot be restored to its original cost.
3. Depreciation is a gradual and containing process because the value of the assets will decline either by their constant use or obsolescence due to expiry of time.
4. Depreciation is not the process of valuation of asset but process of allocation of the cost of an asset to its effective span of life.
5. It diverse as only the book value of the asset, not the market value.
6. It is non-cash expenses. It does not involve any cash outflow.

Causes of Depreciation : Main causes of depreciation are as follows:

(1) By constant use due to the constant use of fixed assets in business operation wear and tear arise in them which results in the reduction of their values.
(2) By expiry of time- The value of majority of assets decreases with the passage of time even if they are not being put to use in the business. Natural form such on rain, winds etc. contribute to the deterioration of their values.
(3) By accident sometimes a machine may be destroyed due to fire, earth quick flood etc. or a vehicle may be damaged due to accident.
(4) By permanent fall in market price through the fluctuations in the market value of fixed asset are not recorded because such assets are not meant for resale but for use in the business, sometimes the fall in the value of certain fixed assets is treated as depreciation such as permanent fall in the value of investments.

CENTRAL BOARD OF SECONDARY EDUCATION

STUDY MATERIAL: APPLIED MATHEMATICS {Class XI}

UNIT 5: SET THEORY

UNIT -5

SET THEORY

Sets and their representations: A given collection of objects is said to be well defined, if we can definitely say whethera given particular object belongs to the collection or not.

Set: A well-defined collection of objects is called a Set. The objects in a set are its numbers or elements or points.

Sets are usually demoted by capital letters A,B,C,X,Y,Z etc. The elements of a set are represented by small letter a,b,c,x,y,z etc.

It is a an elements of a set A, we write a∈A which means that a belongs to A or that a is an elements of A. if a does not belongs to A, we write a∉A.

Illustrations:

(i) The collection of all vowels in the English alphabets contains five elements namely a,e,I,o,u.

So this collection is well defined and therefore, it is a Set.

(ii) The collection of all odd natural numbers less than 10 contains the numbers 1,3,5,7,9.

So this collection is well defined and therefore, it is a Set.

(iii) The collection of all rivers of India is clearly well defined and therefore, it is a Set. Clearly rivers Ganga belongs to this set while rivers Nile does not belong to it.

There are two methods of representing a set:

(i) Roster form or Tabular form- under this method we list all the members of the set within brackets{} and separates them by commas.

Note:- In roster form , the order in which the elements are listed is immaterial.

Ex: Write each of the following sets in the roster form:

(i) A= Set of all factor of 24.
(ii) B= Set of all integers between- $\frac{-3}{2}$ and $\frac{11}{2}$.
(iii) C= Set of all letters in the word 'TRIGONOMETRY'

(iv) D= Set of all months having 30 days.

Solution:

(i) All factors of 24 are 1,2,3,4,6,8,12,24. A= {1,2,3,4,6,8,12,24}.

(ii) All integers between $\frac{-3}{2}$ and $\frac{11}{2}$ are – 1,0,1,2,3,4,5. ∴ B= {-1,0,1,2,3,4,5}.

(iii) If may be noted here that the repeated letters are taken only once each. ∴ C= {T,R,I,G,O,N,M,E,Y}.

(iv) We know that the months having 30 days are April, June, September, and November. ∴ D= {April, June, September, November}.

Note:- we denote the set of all natural numbers all integers, all rational numbers and all real numbers by N,Z,Q and R respectively.

(ii) Set Builder form: under this method of describing a set we list the property or properties satisfied by all the elements of the set.

We write , {x:x has properties P}, we read it as, 'the set of all those are such that each x satisfies properties P'.

Ex: Write the set A= {1,2,3,4,5,6,7} in the set builder form.

Solution: Clearly, A= set of all natural numbers less then 8.

Thus in the set builder form we write it as A={x:x€N and x <8}.

Ex2: Write the st A= {2,4,8,16,32} in the set- builderform.

Solution: Clearly A ={$2^1, 2^2, 2^3, 2^4, 2^5$}.

Thus, in the set builder form we write it as A= {x:x =$2^n$where n€N and 1≤n ≤5}

Ex3: Write the set A ={1,4,9,16,25,….} in set builder form.

Solution: We may write the set A as A= {x: x is the square of a natural numbers} Alternatively, we can write A= { x: x=n^2, where n∈N}.

Empty set: A set containing no elements at all is called the empty set or the null set or the void set denoted by ∅ or {}.

Ex 1:{x : x ∈ N and 2 < x < 3 } = ∅

Ex 2:{x : x ∈ R and x^2 = -1} =∅

Ex 3:{x :x^2 = 4, x is odd } =∅

SINGLE TO N SET: A set containing exactly one element is called a singleton set.

Ex 1:{0} is a singleton set whose only element is 0.

Ex 2: { x : x ∈ N and x^2= 4 } ={2} ,since 2 is the only natural number whose square is 4.

Finite and Infinite sets: A set which is empty or consists of a definite number of elements is called finite otherwise, the set is called infinite.

The number of distinct elements contained in a finite set A is denoted by n (A).

Ex: Let w be the set of the days of the week. Thus w is finite.

Ex: Let S be the set of solution of the equation x^2- 16= 0, thus S is finite.

Note: All infinite sets cannot be described in roster form because the number of elements of such a set is not finite. So we represent some finite set in the roster form by writing a few elements which clearly indicate the structure of the set followed bythreat dots.

For example 1 : { 1,2,3,⋯} is the set of natural numbers.

Ex 2: {⋯,-3,-2,-1,0,1,2,3…} is the set of integers.

Equal sets: Two sets A and B are said to be equal if they have exactly the same elements and we write A = B otherwise, the sets are said to be unequal and we write A≠ B.

Ex 1 : Let A = { 1,2,3,4} and B= {3,1,4,2}. Then A=B.

Ex 2: A set does not change if one or more elements of the set are repeated, the set A= {1,2,3} and B= {2,2,1,3,3} are equal, since each elements of A is in B and vice- versa.

Equivalent sets: Two finite sets A and B are said to be equivalent, if n (A)=n(B).

Equal sets are always equivalent, but equivalent sets need not be equal.

Ex: Let A ={1,3,5} and B= {2,4,6} thus, n(A)= n (B)= 3.

So, A and B are equivalent clearly A≠ B. Hence A and B are equivalent sets but not equal.

Sub set: A set A is said to be a subset of set B if every elements of A is also an elements of B and we write A ⊑ B.

Super set: If A ⊆ B then B is called a super set of A and we write B ⊇ A.

Proper Subset: If A⊆ B and A≠B thus A is called a Proper subset of B and we write ABC.

In other words A C B if whenever a∈ A, thena∈B. It is often convenient to use the symbol "⇒" which means implies. Using this symbol we can write the definition of subset as follows: A⊂B if a∈ A⇒a∈B.

If A is not subset of B we write A⊄B. We may note that for A to be a subset of B, all that is needed is that every elements of A is in B. It is possible that every elements of B may or may not be in A. If it so happens that every element of B is also in A, then we shall also have B C A. In this case A and B are the same sets so that we have A C B and B C A⟺A=B. Where "⟺" is a symbol for two way implications and is usually read as if and only if (briefly written as "iff").

It follows from the above definition that every set A is a subset of itself i.e., A⊑ A. Since the empty set ∅ has no elements we agree to say that ∅ is a subset of every set.

Note: The total number of subset of a set containing n elements is 2^n.

Subssets of Set of real numbers:

The set of natural numbers N={1,2,3,4,5,…}

The set of Integers Z={…,-3,-2,-1,0,1,2,3,…}

The set of rational numbers Q= {x: x =$\frac{p}{z}$, p,q,∈ Z and q≠0}

The set of irrational number T= { x : x ∈ R and x∉Q}, some of the obvious relations among these subsets are: N ⊂ Z ⊂ Q, Q ⊂ R, T⊂ R, N⊄T.

Intervals as subsets of R: Let a,b∈ R and a <b. then the set of real numbers {y:a<y<b} is called an open interval and is denoted by (a,b). All the points between a and b belongs to the open interval. (a,b) but a ,b themselves do not belong to this interval.

The interval which contains the end points also is called closed interval and is denoted by [a,b] thus [a,b]= {x:a≤ x≤ b}.

We can also have intervals closed at one end and open at this other i.e.,

(a,b) or (a,b[= {x:a≤x< b} is an right half open interval. (a,b] or]a,b] = {x:a<x≤ b} is an left half open interval.

The set [o,∞] defines the set of non-negatives real number, while set (-∞ −○) defins the set of negative real numbers. The set (-∞, ∞) describe the set of real numbers in relation to a line extending from -∞ to ∞ on real number line various types of intervals described above as subsets of R, are shown below:

Hence, we note that an interval contains infinitely many points.

Length of Interval: The length of each of the intervals [a,b], [a,b), and (a,b] is (b-a).

Power set : The collection of all subsets of a set A is called the power set of A. It is denoted by P(A). In P(A), every elements is a set.

In general, if A is set with n (A)= m, thus n [P(A)]= z^m.

Ex: If A= {1,2}, thus P(A) = $\{$Ø,{1^3}, {2^3},{1,2}$\}$, Also note that n [P(A)] =2^2 =4.

Universal Set: Usually in a particular context we have to deal with the elements and subsets of a basic set which is relevant to that particular context the basic set is called the universal set.

The universal set is usually denoted by U and all its subsets by the letters A, B, C etc.
For example, for the set of all integers the universal set can be the set of rational numbers or for that matter the set R of real numbers.
Ex: When we discuss sets of lines, triangles or circles in two dimensional geometry the plane in which these lines, triangles or circles lie is the universal set.

Ex: Write down power set of Ø.

Solution: Ø has only one sub- set namely Ø,∴ P(Ø)={Ø}

Equal Sets: Two sets A and B are said to be equal , if every element of A is in B and every element of B is in A and we write A=B. we can write it as A=B⟺A⊆B and B⊆A.

Ex: Let A= $\{$1,{2},{3,4},5$\}$, which of the following are incorrect statements? Rectify each:

(i) 2∈A
(ii) {2} CA
(iii) {1,2} CA.
(iv) {3,4} CA
(v) {1,5} CA
(vi) {Ø} CA
(vii) ICA
(viii) {1,2,3,4}CA

Solution: Clearly A contains four elements namely 1,{2}, {3,4} and 5.

- **(i)** 2∈A is incorrect. The correct statement would be {2}∈A.
- **(ii)** {2} C A is incorrect. The correct statement is {{2}}CA.
- **(iii)** Clearly,2∉ A and therefore {1,2} C A is incorrect. The correct statement would be {1,{2}}CA.
- **(iv)** Clearly, {3,4} is an elements of A. So {3,4} C A is incorrect and {{3,4}} C A is correct.
- **(v)** Since 1 and 5 are both elements of A, So {1,5} C A is correct.
- **(vi)** Since ∅ ∉A , so {∅} C A is incorrect while ∅ C A is correct.
- **(vii)** Since 1∈ A, so I C A is incorrect and therefore,{1} C A is correct .
- **(viii)** Since 2∉ A and 3∉ A, so {1,2,3,4} C A is incorrect The correct statement would be {1,{2}{3,4} C A .

Operation of sets:

Union of sets: The union of two sets A and B denoted by A ∪ B, is the set of all those elements which are either in A or B or in both A and B.

Thus A ∪ B= { x:x ∈ A or x ∈ B} , ∴ x∈ A∪B⇔ x∈A or x ∈ B.

Ex 1: Let A ={ x : x is a positive integer} and let B= { x : x is a negative integer} find A ∪ B.

Solution:Clearly A ∪ B = {x : x is an integer and x ≠○}.

Ex 2: If A ={ x : x =2n +1, n ∈ z} and B= {x : x =2n, n ∈z} thus find A ∪ B.

Solution :We have A∪ B={ x : x is an odd integer } ∪ {x : x is an even integer}

= {x : x is an integer} = Z

Intersection of Sets: The intersection of two sets A and B, denoted by A ∩ B is the set of all elements which are common to both A and B.

Thus A ∩ B = { x : x ∈ A and x ∈ B}
∴ x ∈ A ∩ B ⇔ x ∈ A and x ∈ B,
x∉ A ∩ B ⟹ x ∉ A or x ∉ B.

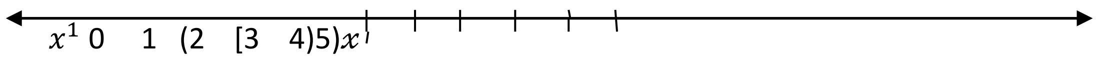
Ex1: If A = { x : x ∈ N, x is a factor of 12} and B = {x : x ∈ N, x is a factor of 18} find A ∩ B.

Solution :We have A = {x : x ∈ N, x is a factor of 12} = {1,2,3,4,6,12}
B = {x : x ∈ N, x is a factor of 18}
∴ A∩ B = {1,2,3,4,6,12} ∩ {1,2,3,4,6,9,18} = {1,2,3,6}
Ex2 :If A =(2,4) and B =[3,5) find A ∩B.

Solution : we have A =(2,4) ={x : x ∈ R, 2< x <4}

B =[3,5) = {x : x ∈R, 3≤ x < 5}

Clearly, A ∩B ={x : x ∈R, 3 ≤x < 4} =[3,4)



Disjoint sets: Two sets A and B are said to be disjoint if A ∩ B = ∅

Ex: If A = {1,3,5,7,9}, B= {2,4,68} find A ∩ B.

Solution : we have A ∩ B = {1,3,5,7,9,} ∩ {2,4,6,8}=∅.

Differences of sets: for any sets A and B, their difference (A-B) is defined as (A-B) ={x : x ∈ A and x ∉ B}

Thus x ∈ B (A – B) ⇒ x ∈ A and x ∉ B

Ex: If A = {x : x∈ N, x is a factor of 6} and

B = {x : x ∈ N, x is a factor of 8 } then find (i) A – B , (ii) B – A.

Solution : We have A = {x : x ∈ N, x is a factor of 6} = {1,2,3,6} and

B = {x : x ∈ N, x is a factor of 8} = { 1,2,4,8}

(i) A – B = {1,2,3,6} – {1,2,4,8} = {3,6}
(ii) B – A = {1,2,4,8} - {1,2,3,6} = {4,8}

Symmetric difference of two sets : The Symmetric difference of two sets A and B, denoted by A Δ B, is defined as : A Δ B = (A – B) ∪ (B-A).

Ex: Let A = {a,b,c,d} and B= {b,d,f,g,} find A Δ B.

Solution : We have

(A-B)= {a,b,c,d} – {b,d,f,g,}

= {a,c}

(B-A)= {b,d,f,g,} – {a,b,c,d}

= {f,g}

∴ A Δ B = (A – B) ∪ (B-A)= {a,c} ∪ {f,g}

={a,c,f,g}

Compliment of a set: Let ∪ be the universal set and let $A \sqsubset U$. Then, the compliment of A denoted by À or (U-A), is defined as

À= {x∈U : x ∉ A }

Clealy x ∈ À⇒ x ∉ A.

<u>Ex</u>: If U= {1,2,3,4,5,6,7,8} and A ={2,4,6,8} find (i) À (ii)(À)´

<u>Solution</u>: we have

(i) À= U – A
=\{1,2,3,4,5,6,7,8\} – \{2,4,6,8\}
=\{1,3,5,7\}

(ii) (À)' =U –À

=\{1,2,3,4,5,6,7,8\} – \{1,3,5,7\}

=\{2,4,6,8\}

= A

<u>Ex</u>: If A CU proves that

(i) $U^1 = \emptyset$ (ii) $\emptyset´ = U$, (iii) (A)' = A , (iv) AUA'=U, (v) A ∧ A'=∅

<u>Solution:</u> We have

(i) U'= U – U= {x ∈ U : x ∉ U} = ∅
(ii) $\emptyset^1$= {x ∈ U: x ∉∅} = U
(iii) (A´)´= {x ∈ U : x ∉ A´} = {x ∈ U: x ∈ A} = A
(iv) A U A´ = {x ∈ U : x ∈ A U A´}
= {x∈U : x ∈ A or x ∈ A´}
= {x ∈U : x ∈ A or x ∉ A} = U
(v) A ∩A´ = {x ∈ U : x ∈ A ∧A'}
={ x∈ U : x ∈ A and x ∈ A´}
= {x ∈U : x ∈ A and x ∉ A}
= ∅

Laws of operations on sets:

Theorem 1:

Idempotent Laws:

(i) $A \cup A = A$, (ii) $A \cap A = A$

Theorem 2:

Identity Laws: for any set A,

(i) $A \cup \emptyset = A$, (ii) $A \cap U = A$

Note: $\emptyset$ and U are the identity elements for union and intersection of sets respectively.

Theorem 3:

Commutative Laws: for any two sets A and B,

(i) $A \cup B = B \cup A$
(ii) $A \cap B = B \cap A$.

Theorem 4:

Associative Laws: for any sets A, B, C.

(i) $(A \cup B) \cup C = A \cup (B \cup C)$
(ii) $(A \cap B) \cap C = A \cap (B \cap C)$

Theorem 5:

Distribution Laws: for any three sets A, B, C.

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

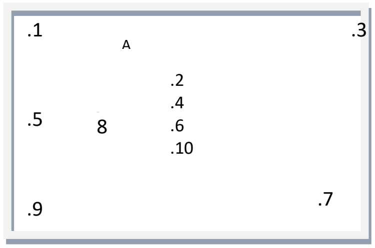
Theorem 6:

Demorgans Laws: for any two sets A and B.

(i) $(A \cup B)^1 = A^1 \cap B^1$
(ii) $(A \cap B)^1 = A^1 \cup B^1$

VENN DIAGRAMS: Most of the relationships between sets can be represented by means of diagrams which are known as venn diagrams. These diagrams consists of rectangles and closed curves usually circles. The universal set is represented usually by a rectangle and its subset by circles.

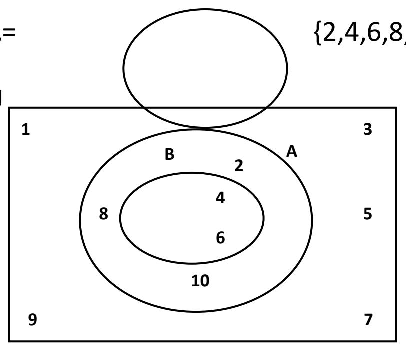
In venn diagrams, the elements of the sets are written in their respective circles.



U= {1,2,3,…,10}

A= {2,4,6,8,10}

U



U= {1,2,3….10}

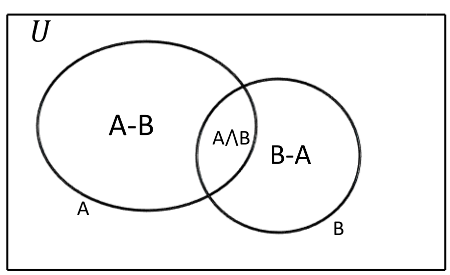
A= {2,4,6,8,10 }

B= {4,6},

A and B are subsets and also B C A.

<u>VENN DIAGRAMS IN DIFFERENT SITUATIONS:</u>

1. When two intersection subsets of U are given:



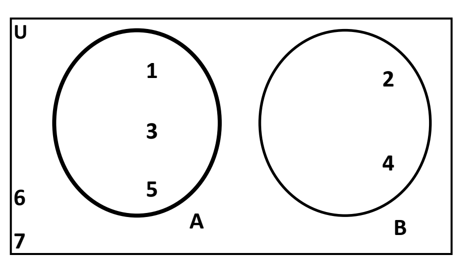
2. <u>When two disjoint subsets of a set be given :</u>

U= {1,2,34,5,6,7}

A= {1,3,5}

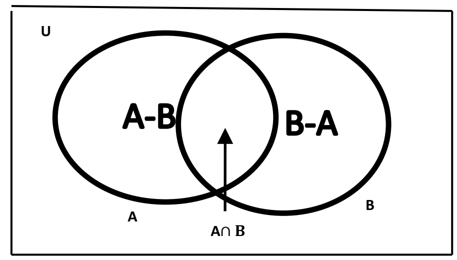
B= {2,4}

Clearly A ∩ B = ∅, (A-B)={1,3,5}=A



(B-A) = {2,4} =B

A^1= {2,4,6,7} and B^1 ={1,3,5,6,7}



<u>Practical problems on union and intersection of two</u>:

<u>Sets</u>: LetA and B be finite sets. If $A \cap B=\emptyset$, then

(i) n in general, if A and B are finite sets, thus (AUB) =n (A) +n (B)

(ii) n (AUB) = n (A) + n (B) n(A∧B)

Note that the sets (A – B), A∩ B and (B∩A) are disjoint and their union is A ∪ B. therefore n (A∪ B) = n (A – B) + n (A ∩ B) + n (B – A)
= n (A – B) + n (A∩ B) + n (B- A)+ n (A∩B) – n (A∩B)

= n (A) +n (B) – n (A∩B)

(iii) If A,B and C are finite sets, then n (AUBUC) = n (A) + n (B) + n (C) – n (A∩B) – n (B∩C) – n (A∩C) + n (A∩B∩C)

<u>Ex 1:</u>In a school there are 20 teachers who teach Mathematics or Physics or these, 12 teach Mathematics and 4 teach both Physics and Mathematics. How many teach Physics?

Solution: Let M denote the set of teachers who teach mathematics and P denote the set of teachers who teach Physics. In the statement of the problem the word 'or' gives us a clue of union and the word 'and' gives us a clue of intersection. We therefore have.
n (MUP) = 20, n (M) =12 and n (M∩P) =4 we wish to determines n (P).. using the result.
n (MUP) = n (M) → n (P) – n (M∩P) we obtain. 20= 12+n(P)- 4,
thus n (P) =12.
Hence 12 teachers teach Physics.

Ex 2: In a survey of 400students in a school 100 were listed as taking apple juice ,150as taking as orange juice 75 were listed as taking both apple as well as orange juice. Find how many students were taking neither apple juice nor orange juice.

Solution : Let U denote the set of survived students and A dente the set of students taking apple juice and B denote the set of students taking orange juice. Thus

$n(U) = 400$, $n(A) = 100$ $n(B) = 150$ and $n(A \cap B) = 75$

Now $n(A^1 \cap B^1) = n(A \cup B)^1$

$= n(U) - n(A \cup B)$

$= n(U) - n(A) - n(B) + n(A \cap B)$

$= 400 - 100 - 150 + 75$

$= 225$

Hence 225 students were taking neither apple juice nor orange juice.

Ex 3: In a group of 850 person 600 can speak Hindi and 340 can speak Tamil. Find

(i) How many can speak both Hindi and Tamil.
(ii) How many can speak Hindi only.
(iii) How many can speak Tamil only.

Solution : Let A = set of person who can speak Hindi and B =set of person who can speak Tamil.

$\therefore n(A) = 600$, $n(B) = 340$ and $n(A \cup B) = 850$.

(i) Set of person who can speak both Hindi and Tamil = $A \cap B$.

Now, $n(A \cap B) = n(A) + n(B) - n(A \cup B)$

$= 600 + 340 - 850$

$= 90.$

Thus 90 person can speak both Hindi and Tamil.

(ii) Set of person who can speak Hindi only = $(A-B)$.

Now , $n(A-B) + n(A \cap B) = n(A)$

$\Rightarrow n(A-B) = n(A) - n(A \cap B)$

$= 600 - 90$

$= 510.$

Thus 510 person can speak Hindi only

(iii) Set of person who can speak Tamil only = $(B - A)$

Now , n (B-A) + n (A∩B)= n (B)

⟹n (B-A) = n (B) – n (A∩B)

=340 – 90

=250

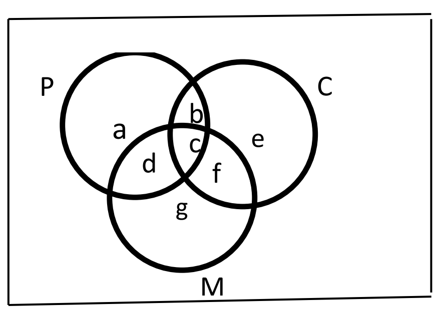
Hence, 250 person can speak Tamil only .

Ex 4 : In a survey of 25 students , it was found that 12 have taken Physics 11 have taken Chemistry and 15 have taken Mathematics 4 have taken Physics & chemistry and 9 have taken Physics and Mathematics 5 have taken Chemistry and Mathematics while 3 have taken all the three subjects. Find the numbers of students who have taken.

(i) Physics only.
(ii) Chemistry only.
(iii) Mathematics only.
(iv) Physics and Chemistry but not Mathematics.
(v) Physics and Mathematics but not Chemistry.
(vi) Only one of the subjects.
(vii) At least one of the three subjects.
(viii) None of the three subjects.

Solution: Let P ,C and M be the sets of students who have taken Physics ,Chemistry and Mathematics respectively.

Let a, b, c, d, e, f and g denote the numbers of students in the respective regions. As per data given, we have:



a+b+c+d = 12

b+c+e+f = 11

c+d+f+g = 15

b + c = 4

c + d = 9

c + f = 5

c = 3

from these equation we get: $c = 3$, $f= 2$, $d = 6$, $b = 1$.

Now $c + d + f + g = 15 \Longrightarrow 3 + 6 +2 + g =15 \Longrightarrow g = 4$,

$b + c + e + f = 11 \Longrightarrow 1 + 3 + e + 2 = 11 \Longrightarrow e = 5$

$a + b + c + d = 12 \Longrightarrow a + 1 + 3 + 6 = 12 \Longrightarrow a = 2$

$\therefore$ a= 2, b= 1, c= 3, d= 6, e= 5, f=2 and g= 4, so we have:

(i) Number of students who offered Physics only a= 2.

(ii) Number of students who offered Chemistry only, e= 5.

(iii) Number of students who offered Mathematics only, g= 4.

(iv) Number of students who offered Physics and Chemistry but not Mathematics, b= 1.

(v) Number of students who offered Physics and Mathematics but not Chemistry, d= 6.

(vi) Number of students who offered only one of the given subjects (a+e+g)=(2+5+4)=11.

(vii) Number of students who offered at least one of the given subjects = (a+b+c+d+e+f+g) = (2+1+3+6+5+2+4) =23.

(viii) Number of students who offered none of the three given subjects =(25 – 23) =2.

CENTRAL BOARD OF SECONDARY EDUCATION

STUDY MATERIAL: APPLIED MATHEMATICS {Class XI}

UNIT 6: RELATIONS & FUNCTIONS

UNIT – 6

Relations and Functions

Ordered Pair: Two numbers a and b listed in a specific order and enclosed in parentheses form an ordered pair (a,b). In the ordered pair (a,b), we call a as first number and b as second number.

Thus $(a,b) \neq (b,a)$

Equality of two ordered pairs: we have $(a,b) = (c,d) \Leftrightarrow a = c$ and $b = d$.

Cartesian products of two sets: Let A and B be two non-empty sets. Then, the Cartesian product of A and B is the set denoted by (A X B), consisting of all ordered pairs (a,b) such that $a \in A$ and $b \in B$.

$\therefore (A \times B) = \{(a,b) : a \in A \text{ and } b \in B\}$

If $A = \emptyset$ or $B = \emptyset$, we define $A \times B = \emptyset$.

Remarks:

(i) If n (A) = p and n (B) = q then n (AXB) = pq and n (BXA) = pq.
(ii) If at least one of A and B is infinite thus (AxB) is infinite and (BXA) is infinite.

Ex 1: If $(x + 1, y - 2) = (3,1)$, find the value of x and y.

Solution: Since the ordered pairs are equel, the corresponding elements are equal. Therefore

$X + 1 = 3$ and $y - 2 = 1$

Solving we get : $x = 2$ and $y = 3$.

Ex 2 : If AXB = $\{(p,q), (p,r), (m,q), (m,r)\}$ find A and B.

Solution : A = set of first elements = {p,m}

B = set of second elements = {q,r}

Ordered Triplet: Three numbers a , b and c listed in a specific order and enclosed in parent theses form an ordered triplet (a, b, c).

Thus $(1,2,3) \neq (2,1,3) \neq (3,2,1)$ etc.

For any non empty set A, we define

$(A \times A \times A) = \{(a, b, c) : a, b, c \in A\}$

EX: If P = {1,2}, from the set P x P x P.

Solution : If P = {1,2} thus

$P \times P = \{1,2\} \times \{1,2\}$

$= \{(1,1), (1,2), (2,1), (2.2)\}$

$P \times P \times P = (P \times P) \times P$

$= \{(1,1), (1,2), (2,1), (2,2)\} \times \{1,2\}$

$= \{(1,1,1), (1,1,2), (1,2,1), (1,2,2), (2,1,1), (2,1,2), (2,2,1), (2,2,2)\}$

Ex2: If R is the set of all real numbers, what do the Cartesian products R x R and R x R x R represent?

Solution: The Cartesian product R x R represents the set $R \times R = \{(x,y) : x,y \in R\}$ which represents the coordinates of all the points in the two dimensional space and the Cartesian product $R \times R \times R = \{(x, y, z) : x, y, z \in R\}$ which represents the coordinates of all the points in three dimensional space.

Relations: A relation R from a non-empty set A to a non-empty set B is a subset of the Cartesian product A x B.

Thus R is a relation from A to B $\Leftrightarrow R \subseteq (A \times B)$. If $(a,b) \in R$ thus we say that 'a' is related to 'b' and we write , a R b.

Domain, Range and Co-domain of a Relation : Let R be a relation from A to B. Thus $R \subseteq (A \times B)$.

(i) The set of all first coordinates of elements of R is called the domain of R, written as dom (R) .
(ii) The set of all second coordinates of elements of R is called the range of R, denoted by range (R).
(iii) The set B is called the co-domain of R .
Dom (R) = $\{a : (a,b) \in R\}$ and Range (R) = $\{b : (a,b) \in R\}$

Total number of Relation from A to B : Letn(A) = P and n(B) = q, Thus , n(A x B) = pq. We know that every subset ofA x B is a relation from A to B.

∴ Toatal number of sub set of A x B is 2^{pq}.

∴ Total number of relation from A to B = 2^{pq}.

Representation of a Relation: Let A and B be two given sets ,thus a relation R ⊆A x B can be represented in any of the forms given below.

(1) ROSTER FORM: In this form R is given as a set of ordered pairs.

Ex 1 : Let A = { -2 , -1 ,0 , 1, 2} and B = {0, 1, 4, 9}

Let R = $\{$(-2,4), (-1,1), (0,0), (1,1), (2,4)$\}$

(i) Show that R is a relation from A to B.

(ii) Find dom (R), range (R) and co-domain of R.

Solution:

(i) Since R C A x B, so R is a relation from A to B. Note that – 2R4 , -1R_1, 0R0, 1R1 and 2R4.

(ii) Dom R = Set of first coordinate of elements of R = {-2, -1, 0, 1, 2}.

Range (R) = Set of second coordinates of elements of R.

= {0, 1, 4}

Co-domain of R = {0, 1, 4, 9} =B.

(2) SET BUILDER FORM : Under this method for every (a,b) ∈ R, a general relation is being given between a and b. Using this relation, all the elements of R can be obtained.

Ex1: Let A = {1, 2, 3, 5} and B = {4, 6, 9} define a relation from A to B, given by

R =$\{$(a, b) : a ∈ A, b∈ B and (a-b) is odd$\}$

(i) Write R in roster form.

(ii) Find dom (R) and Range (R).

Solution:

(i) Clearly, we have

R = $\{$(1,4) , (1,6) , (2,9) , (3,4) , (3,6) , (5,4) , (5,6)$\}$

(ii) Dom (R) = Set of first coordinates of elements of R = {1, 2, 3, 5}

Range (R) = Set of second coordinates of elements of R = {4, 6, 9}

(3) **Arrow Diagram :** Let R be a relation from A to B. first we draw two bounded figures to represent A and B respectively. We mark the elements of A and B in these figures. For each $(a,b) \in R$, we draw an arrow from a to b.

Ex: Let A = {1, 2, 3, 4, 5}. Define a relation R from A to A by R = {(x,y) : y = 2x – 3}.

(i) Depict R using an arrowdiagram.

(ii) Find dom (R) and range (R).

Solution : $x = 1 \Rightarrow y = (2 - 3) = -1 \notin A$

$x = 2 \Rightarrow y = (4 - 3) = 1$

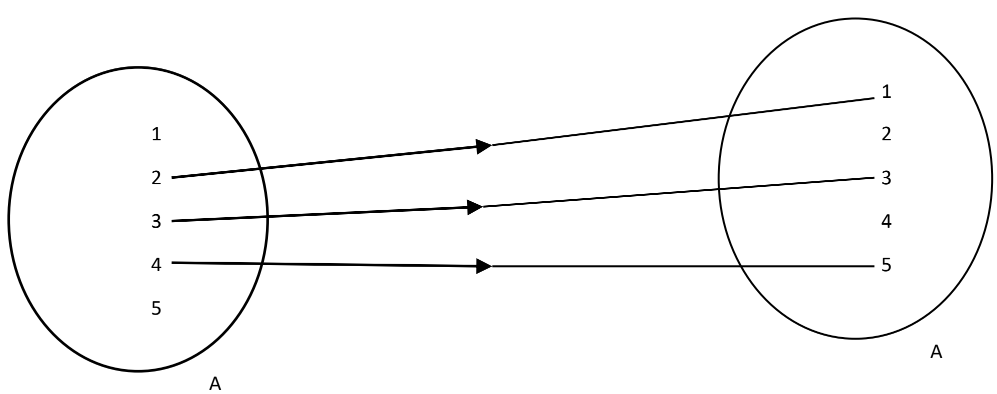
$x = 3 \Rightarrow y = (6 - 3) = 3$

$x = 4 \Rightarrow y = (8 - 3) = 5$

$x = 5 \Rightarrow y = (10 - 3) = 7 \notin A$

$\therefore R = \{(2, 1), (3, 3), (4, 5)\}$

(i) We may depict it by arrow diagram, as



(ii) We have dom (R) = {2 , 3, 4} and range (R) = {1,3,5}

Functions: we can visualize a function as a rule which produces new elements out of some given elements. There are many terms such as ‘map’ or ‘mapping’ used to denote a function.

Definition: A relation f from a set A to a set B is said to be a function if every element of set A has one and only one image in set B.

In other words , a function f is a relation from a non-empty set A to a non empty set B such that the domain of f is A and no two distinct ordered pairs in f have the same first element.

If f is a function from A to B and (a,b) ∈ f, then f (a) = b, where b is called the image of a under f and a is called the pre image of b under f. The function f from A to B is denoted by f: A → B.

Dom (f) = A and range (f) ⊆ B, Also B is called the co-domain of f.

Ex1 : Let x = {1, 2, 3 4} and y = {1, 4, 9, 16, 25}

Let f = { (x, y) : x ∈ X , y ∈ y and y = x^2}

(i) Show that f is a function from X to Y. find its domain and range.
(ii) Draw a pictorial representation of the above functions.
(iii) If A = {2, 3, 4} , find f (A).

Solution :

(i) We have f = {(x,y) : x ∈ X, y ∈ y and y = x^2}

Giving different values to x from the set X and gifting corresponding value of y = x^2, we get

f = $\{(1, 1), (2, 4), (3, 9), (4, 16)\}$

Clearly, every element in x has a unique image in y.

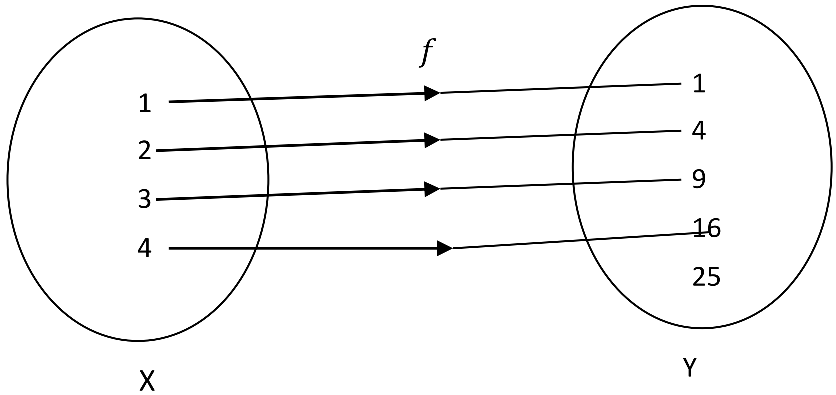
Hence, f is a function from X to Y.

Dom (f) = {1, 2, 3, 4} = X

Range (f) = {1, 4, 9, 16} ⊂Y.

Clearly, 25 ∈ Y does not have its pre-image in X.

(ii) A pictorial representation of the above mapping f is given by



(iii) Now, let A = { 2, 3, 4}. Thus

f (2) = 2^2 = 4 , f (X 3^2 = 9 and f (4) = 4^2 = 16

$\therefore f$ (A) = {f(x) : x ∈ A} = {4, 9, 16}

<u>Ex</u> : Examine each of the following relation give below and state in each case giving reasons whether it is a function or not?

(i) R = $\{(2, 1), (3, 1), (4, 2)\}$

(ii) $\{(2, 2), (2,4), (3, 3), (4, 4)\}$

(iii) R = $\{(1,2), (2,3), (3, 4), (4, 5), (5, 6), (6, 7)\}$

<u>Solution</u> :

(i) Since 2, 3, 4 are the elements of domain of R having their unique images, this relation R is a function.

(ii) Since the same first elements 2 corresponds to two different images 2 and 4, this relation is not a function.

(iii) Since every element has one and only one image, this relation is a function.

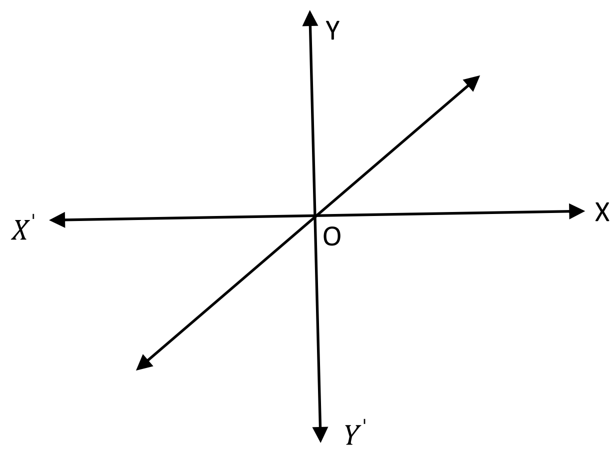
Definition: A function which has either R or one of its subsets as its range is called a real valued function. Further, if its domain is also either R or a subset of R. It is called a real function.

Some functions and their graphs:

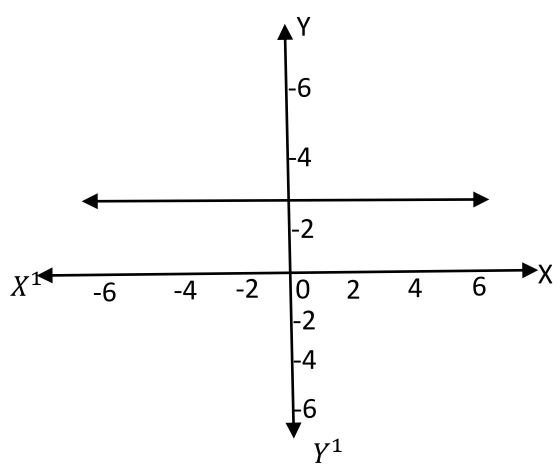
(i) **Identity function**: Let R be the set of real numbers. Define the real valued function.

f : R → R by y = f (x) = x for each x ∈ R.

Such a function is called the identity function. Here the domain and range of f are R. The graph is a straight line. It passes through the origin.



(ii) **Constant functions**: Define the function: R → R by y = f (x) = c, x ∈ R where C is a constant and each x ∈ R. Hence domain of f is R and its range is {c}. The graph is a line parlor to x – axis.



f (x) = 3

(iii) **Polynomial function**: A function: R → R is said to be polynomial function if for each x in R, y = f (x) = ao + a1x + a2 x^2 +_ _ _ _+ an x^n, where n is a non – negative integer and ao, a1, a2, ……..,an ∈ R.

Ex : Define the function f : R → R by y = f (x) = x^2, x ∈ R. complete the table given below by using the definition. What is the domain and range of this function ? Draw the graph of f.

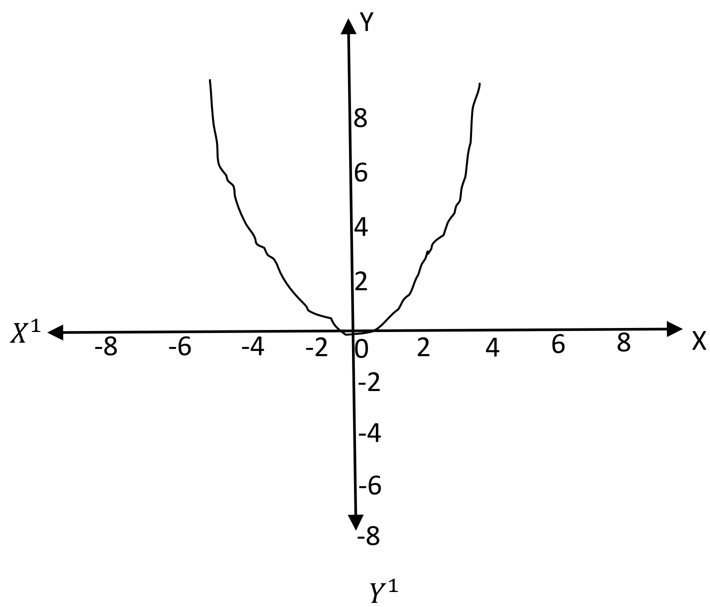
x	-4	-3	-2	-1	0	1	2	3	4
y = f(x) =x^2									

Solution: The complete table is given below:

x	-4	-3	-2	-1	0	1	2	3	4
y = f(x) =x^2	16	9	4	1	0	1	4	9	16

Domain of f = { x : x ∈ R}, Range of f = {x : x ≥ 0 , x ∈ R}

The graph of f is given by



f (x) = x^2

(iv) **Rational functions**: Rational function of the type $\frac{f\text{ (x)}}{g\text{ (x)}}$, where f (x) and g (x) are polynomial functions of x defined in a domain where of (x) $\neq 0$.

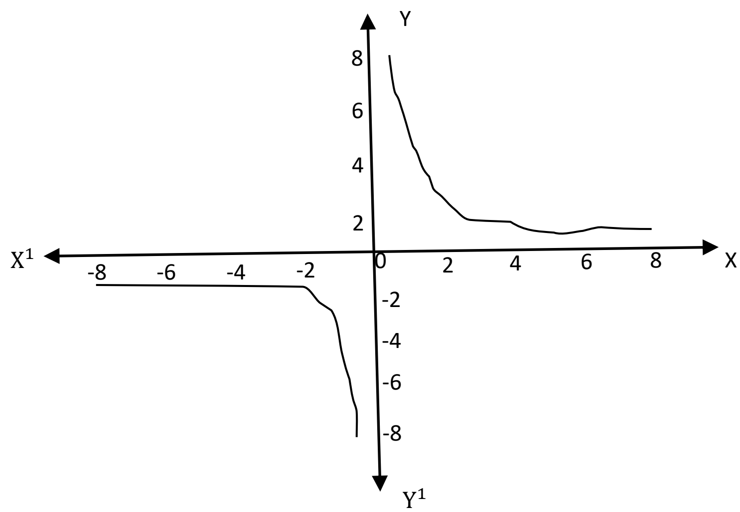
Ex: Define the real valued function: R – {0} → R defined by f (x) = $\frac{1}{x}$, x ∈ R – {0} , complete the table given below using this definition. What is the domain and range of this function?

x	-2	-1.5	-1	-0.5	0.25	0.5	1	1.5	2
y =$\frac{1}{x}$	..	..	..	..	..	..	..	..	..

Solution : The complete table is given by:

x	-2	-1.5	-1	-0.5	0.25	0.5	1	1.5	2
y =$\frac{1}{x}$	-0.5	-0.67	-1	-2	4	2	1	0.67	0.5

The domain is all real numbers except 0 and its range is also all real numbers except 0. The graph of f is given by:

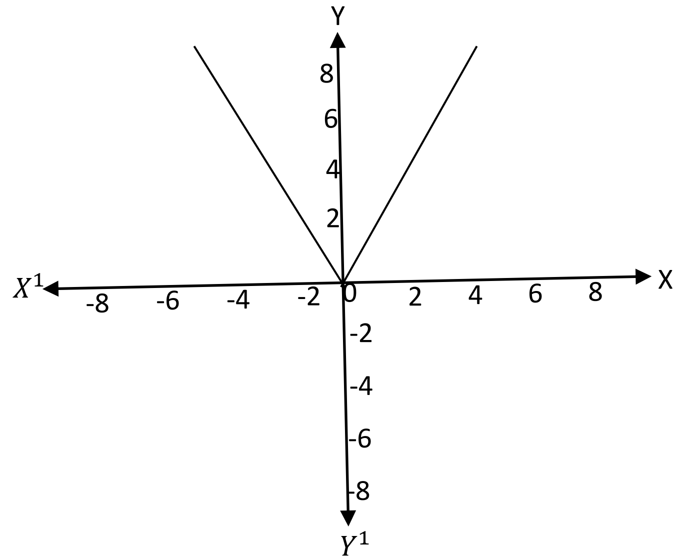


f (x) = $\frac{1}{x}$.

(v) **The modulus function**: The function f : R → R defined by f (x) = |x| for each x ∈ R is called modulus function. For each non – negative value of x ,f (x) is equal to x. But for negative value of x, the value of f (x) is the negative of the value of x, i.e. ,

$$f(x) \begin{cases} x, x \geq ,0 \\ -x, x < 0 \end{cases}$$

The graph of the modulus function is given by:

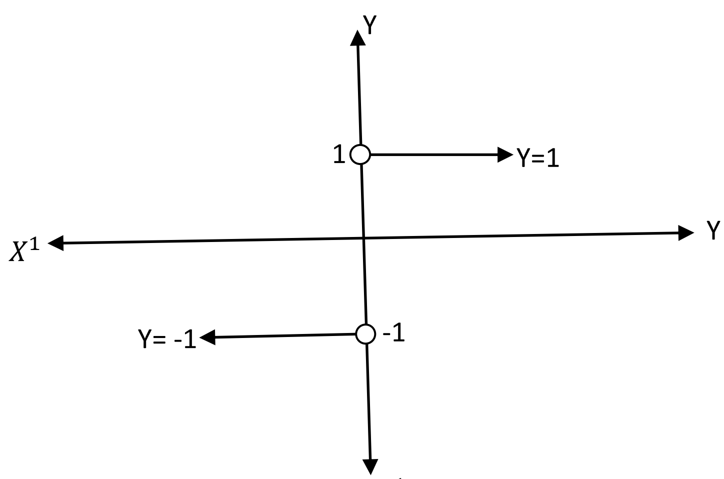


$f(x) = |x|$

(vi) **Signum function**: The function f : R → R defined by

$$f(x) = \begin{cases} 1, if\ x > 0 \\ 0, if\ x = 0 \\ -1, if\ x < 0 \end{cases}$$

Is calledthe Signum function. The domain of the signum function is R and the range is the set { -1, 0, 1}. The graph of the signum function is given by:



$f(x) = \frac{|x|}{x}$, x ≠ 0 and 0 for x = 0

(vii) **Greatest Integer function**: The function f : R $\rightarrow$ R defined by f (x) = [x], x $\in$ R assume the value of the greatest integer, less than or equal to x. Such a function is called the greatest integer function.

From the definition of [x], we can set that

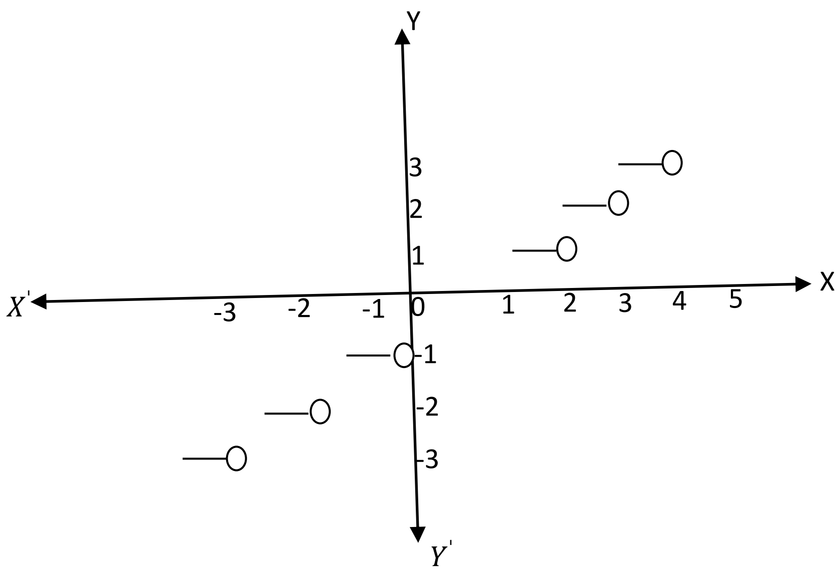
[x] = -1 for -1 $\leq$ x < 0

[x] = 0 for0 $\leq$ x < 1

[x] = 1 for 1 $\leq$ x < 2

[x] = 2 for 2 $\leq$ x < 3 and so m.

The graph of the function is given by:



f(X) = [X]

CENTRAL BOARD OF SECONDARY EDUCATION

STUDY MATERIAL: APPLIED MATHEMATICS {Class XI}

UNIT 7: ALGEBRA

UNIT – 7

Algebra

(A) COMPLEX NUMBER

Concept of Imaginary Numbers and iota: If the square of a given number is negative then such a number is called an imaginary number.

For example, $\sqrt{-1}$, $\sqrt{-2}$ etc are imaginary numbers. We denote $\sqrt{-1}$ by gresse letter iota 'i '. Thus $\sqrt{-4}$ = 2i, $\sqrt{-9}$= 3i etc.

Powers of i : we have $i^{\circ} = 1$, $i^1 = i$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, Let us consider i^n, where n is a positive integer and n>4. On dividing n by 4, let quotient be m and the remainder be r. thus

$$n = 4m + r, \text{ where } 0 \leq r < 4$$

$\therefore \quad i^n = i^{4m+r} = i^{4m} \text{ x } i^r = [i^4)^M \text{ x } i^r = i^r$

Ex:Evaluate:

(i) i^{998} , (ii) i^{-71} , (iii) i^{-1}

Solution:

(i) $i^{998} = i^{4 \, x \, 249 \, + \, 2} = i^2 = -1$

(ii) $i^{-71} = \frac{1}{i^{71}} = \frac{1}{i^{71}} \text{ x } \frac{i}{i} = \frac{i}{i^{72}} = \frac{i}{(i^4)^{18}} = \frac{i}{i} = \text{i}$

(iii) $i^{-1} = \frac{1}{i} = \frac{1}{i} \text{ X } \frac{i^3}{i^3} = \frac{-i}{1} = -i$

Ex: Show that the sum $(1+ i^2 + i^4 + \ldots + i^{2n})$ is 0 when n is odd and 1 when n is even.

Solution: Let $S = 1 + i^2 + i^4 + \ldots + i^{2n}$

This is clearly a GP having (n+1) terms with a = 1 and $r = i^2 = -1$.

$$\therefore S = \frac{^{a}(1-r^{n+1})}{(1-r)} = \frac{1 \, x \, [1-(i^2)^{n+1}]}{(1-i^2)}$$

$$= \frac{\{1-(-1)^{n+1}\}}{1-(-1)} = \frac{\{1-(-1)^{n+1}\}}{2}$$

$$= \begin{cases} \tfrac{1}{2}(1-1) = 0 \text{, when n is odd} \\ \tfrac{1}{2}(1+1) = 1 \text{, when n is even} \end{cases}$$

Note: for any two real numbers a and b , the result $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ is true only when at least one of the given numbers is either zero or positive.

Thus $\sqrt{-2} \times \sqrt{-3} = \sqrt{(-2)\, x\, (-3)} = \sqrt{6}$ is wrong.

In fact $\sqrt{-2} \times \sqrt{-3} = i\sqrt{2} \times i\sqrt{3} = i^2\, x\, \sqrt{6} = -\sqrt{6}$.

Ex : Evaluate :

$\sqrt{-16} + 3\sqrt{-25} + \sqrt{-36} - \sqrt{-625}$

= 4i + 3 x 5 i + 6 i – 25 i

= 0.

Complex Numbers: The numbers of the form (a+ib) where a and b are real numbers and i= $\sqrt{-1}$, are known as complex numbers. The set of all complex numbers is denoted by C

∴ C = $\{$ (a + ib) : a, b ∈ R$\}$

for a complex numbers , z = (a+ib), we have a= real part of z, written as Re (z) and b = imaginary part of z, written as Im (z).

Purely real and Purely Imaginary Numbers :

A complex z is said to be

(i) Purely real , if Im (z) = 0
(ii) Purely imaginary, if Re(z) = 0

Thus each of the numbers 2, -7 ,$\sqrt{3}$ is purely real and , each of the numbers 2i , ($\sqrt{3i}$) ,$\left[\frac{3}{2}i\right]$ is purely imaginary.

Conjugate of a Complex Number: Conjugate of a complex number z = (a + ib) is defined as, $\bar{z}$ = a – ib.

Arithmetic operation on Complex number:

(1) **Addition of two Complex Numbers**: let z_1 = a + ib and z_2 = c + id be any two complex numbers. Then the sum $z_1 + z_2$ is defined as follows :

$z_1 + z_2$ = (a+c) + i (b+d) , which is again a complex number.

The addition of complex numbers satisfy the following properties :

(i) **The closure law** : The sum of two complex numbers is a complex number i.e. $z_1 + z_2$ is a complex number for all complex numbers z_1 and z_2.

(ii) **Commutative law** : For any two complex numbers z_1 and z_2 , $z_1 + z_2 = z_2 + z_1$.

(iii) **The Associative law**: For any three complex numbers z_1 , z_2 , z_3 .

$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$

(iv) **The existence of Additive identity** : There exists the complex number 0 + i0 (denoted as 0), called the additive identity or zero complex number , such that, for every complex number z, z + 0 = z.

(v) **The existence of additive inverse** : To every complex number z = a + i(-b) (denoted as – z) called the additive inverse or negative of z. we observe that z + (-z) = 0.

(2) **Difference of two complex numbers**: Given any two complex numbers z_1 and z_2 is defined as follows:

$z_1 - z_2 = z_1 + (-z_2)$

(3) **Multiplication of two complex numbers** : Let z_1 = a + ib and z_2 = c + id be any two complex numbers. Thus the product $z_1 z_2$ is defined as follows:

$z_1 z_2$ = (ac – bd) + i (ad + bc)

The multiplication of complex numbers possesses the following properties,

(i) The closure law : The product of two complex numbers is a complex number the product $z_1 z_2$ is a complex number for all complex numbers z_1 and z_2.

(ii) The commutative law : For any two complex numbers z_1 and z_2.

$z_1 z_2 = z_2 z_1$

(iii) The Associative law : For any three complex numbers z_1 , z_2 , z_3 ,

$(z_1 z_2) z_3 = z_1 (z_2 z_3)$

(iv) The existence of multiplicative identity : There exists the complex number 1 + i0 (denoted as 1), called the multiplicative identity such that z.1 = z, for every complex number z .

(v) The existence of multiplicative inverse : For every non- zero complex numbers z = a + ib or a + bi (a≠0, b≠0), we have the complex number $\frac{a}{\sqrt{a^2+b^2}}$ + i $\frac{-b}{\sqrt{a^2+b^2}}$ (denoted by $\frac{1}{z}$ or z^{-1}), called the multiplication inverse of z such that z. $\frac{1}{z}$ = 1.

(vi) The distributive law : For any three complex numbers z_1 , z_2 , z_3.

(a) $z_1 \ (z_2 + z_3) = z_1 z_2 + z_1 z_3$.

(b) $(z_1 + z_2)z_3 = z_1 z_3 + z_2 z_3$.

(4) **Division of two complex numbers** : Given two complex numbers z_1 and z_2, where $z_2 \neq 0$, the quotient $\frac{z_1}{z_2} = z_1 \frac{1}{z_2}$

For example , Let z_1 = 6 + 3i and z_2= 2 – i

Then $\frac{z_1}{z_2} = \left\{(6 + 3i) \times \frac{1}{2-i}\right\}$

$$= (6 + 3i) \left\{\frac{2}{2^2 + (-1)^2} + i\frac{-(-1)}{2^2 + (-1)^2}\right\}$$

$$= (6 + 3i)\left(\frac{2+i}{5}\right)$$

$$= \frac{1}{5}\left[12 - 3 + i\,(6 + 6)\right]$$

$$= \frac{1}{5}(9 + 12i).$$

Identities : For any complex numbers z_1 and z_2 , we have the following identities , can be proved to be true for all complex numbers.

(i) $(z_1 + z_2)^2 = {z_1}^2 + {z_2}^2 + 2z_1.z_2$

(ii) $(z_1 - z_2)^2 = {z_1}^2 - 2z_1.\, z_2 + {z_2}^2$

(iii) $(z_1 - z_2)^3 = {z_1}^3 + 3{z_1}^2 z_2 + 3z_1 - {z_2}^2 + {z_2}^3$

(iv) $(z_1 - z_2)^3 = {z_1}^3 - 3{z_1}^2 z_2 + 3z_1 \,.\, {z_2}^2 - {z_2}^3$

(v) ${z_1}^2 - {z_2}^2 = (z_1 + z_2)\,(z_1 - z_2)$.

The Modulus and the conjugate of a complex number : Let z = a + ib be a complex number. Thus the modulus of z , denoted by |z|, is defined to be the non-negative real number $\sqrt{a^2 + b^2}$, i.e.

|z| = $\sqrt{a^2 + b^2}$ and the conjugate of z, denoted as $\bar{z}$, is the complex number a – ib, i.e $\bar{z}$ = a – ib.

For example , |3 +i| = $\sqrt{3^2 + 1^2} = \sqrt{10}$ and $\overline{3 + \imath}$ = 3 – i , $\overline{3\imath - 5}$ = 3i - 5

Observe that the multiplicative inverse of the non-zero complex number z is given by

$$Z^{-1} = \frac{1}{a+ib} = \frac{a-ib}{(a+ib)(a-ib)}$$

$$= \frac{a-ib}{a^2+b^2}$$

$Z^{-1} \quad = \frac{\bar{z}}{|z|^2}$

Or $\qquad Z.\, Z^{-1} = \frac{z\,\bar{z}}{|z|^2}$

Or $\qquad 1 \quad = \frac{z\,\bar{z}}{|z|^2}$

Or $\qquad 1 \quad = |z|^2$

Furthermore, the following results can easily be derived. For any two complex numbers z_1 and z_2 , we have:

(i) $|z_1 z_2| = |z_1|\ |z_2|$

(ii) $|\frac{z_1}{z_2}| = \frac{|Z_1|}{|Z_2|}$ Provided $|Z_2| \neq 0$.

(iii) $\overline{z_1 z_2} = \overline{z_1 z_2}$

(iv) $\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2}$

(v) $(\frac{z_1}{z_2}) = \frac{\overline{z_1}}{\overline{z_2}}$ Provided $z_2 \neq 0$

<u>**Ex**</u>: find the multiplicative inverse of z – 3i.

<u>**Solution**</u>: Let z = 2 – 3i

Thus $\qquad \overline{Z} = 2 + 3i$ and $|Z|^2 = 2^2 + (-3)^2 = 13$

Therefore, the multiplicative inverse of 2 – 3i is given by:

$$Z^{-1} = \frac{\overline{z_1}}{|z_2|^2} = \frac{2+3i}{13} = \frac{2}{13} + \frac{3}{13}i.$$

The above working can be reproduced in the following manner also,

$$Z^{-1} = \frac{1}{2-3\,i} = \frac{2+3\,i}{(2-3\,i)\,(2+3i)}$$

$$= \frac{2+3\,i}{2^2+3^2}$$

$$= \frac{2+3i}{13}$$

$$= \frac{2}{13} + \frac{3}{13}i.$$

Ex: Express the following in the form a + ib

(i) $\frac{5+\sqrt{2}\ i}{1-\sqrt{2}\ i}$ (ii) i^{-35}

Solution: (i) We have $\frac{5+\sqrt{2}\ i}{1-\sqrt{2}\ i} = \frac{5+\sqrt{2}\ i}{1-\sqrt{2}\ i} \times \frac{1+\sqrt{2}\ i}{1+\sqrt{2}\ i}$

$$= \frac{5+5\sqrt{2}\ i + \sqrt{2}\ i - 2}{1-(\sqrt{2}\ i)^2}$$

$$= \frac{3+6\sqrt{2}\ i}{1+2}$$

$$= \frac{3\ (1+2\sqrt{2}\ i)}{3}$$

$$= 1 + 2\sqrt{2}\ i$$

(iii) $i^{-35} = \frac{1}{i^{35}} = \frac{1}{i^{4 \times 8+3}} = \frac{1}{i^3} = \frac{1}{-i}$

$= \frac{1}{-i} \times \frac{i}{i} = \frac{i}{-i^2} = i$

Ex: If (a + b) = $\frac{(x+i)^2}{(2x^2+1)}$ then prove that

$(a^2 + b^2) = \frac{(x^2+1)^2}{(2x^2+1)^2}$

Solution: We have

$$(a + ib) = \frac{(x+i)^2}{(2x^2+1)} = \frac{x^2+i^2+2\ ix}{(2x^2+1)} = \frac{(x^2-1)+i\ (2x)}{2x^2+1}$$

$\Longrightarrow (a + ib) = \frac{x^2-1}{2x^2+1} + i\ .\frac{2x}{2x^2+1}$

$\Longrightarrow |a + ib|^2 = |\frac{x^2-1}{2x^2+1} + i\ .\frac{2x}{2x^2+1}|^2$

$\Longrightarrow a^2 + b^2 = \frac{(x^2-1)^2}{(2x^2+1)^2} + \frac{4x^2}{(2x^2+1)^2}$

$$= \frac{(x^2-1)^2+4x^2}{(2x^2+1)2} = \frac{(x^2+1)2}{(2x^2+1)2}$$

Hence $(a^2 + b^2) = \frac{(x^2+1)^2}{(2x^2+1)^2}$

Ex: If $|z_1| = |z_2| = |z_3| = \ldots\ldots.= |z_n| = 1$ then prove that

$$\left|\frac{1}{z_1} + \frac{1}{z_2} + \ldots\ldots + \frac{1}{z_n}\right| = |z_1 + z_2 + \ldots\ldots + z_n|$$

Solution: We have

$$|z_1| = |z_2| = \ldots\ldots\ldots\ldots = |z_n| = 1$$

$$\Rightarrow |z_1|^2 = |z_2|^2 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots = |z_n|^2 = 1$$

$$\Rightarrow \frac{1}{z_1} = \overline{z_1}, \frac{1}{z_2} = \overline{z_2}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \frac{1}{z_n} = \overline{z_n}$$

$$\Rightarrow z_1\overline{z_1} = 1, z_2\overline{z_2} = 1, \ldots\ldots\ldots\ldots\ldots, z_n\overline{z_n} = 1$$

$$\Rightarrow \left|\frac{1}{z_1} + \frac{1}{z_2} + \ldots\ldots + \frac{1}{z_n}\right| = |\overline{z_1} + \overline{z_2} + \ldots.+ \overline{z_n}|$$

$$\Rightarrow \left|\frac{1}{z_1} + \frac{1}{z_2} + \ldots\ldots + \frac{1}{z_n}\right| = \overline{|z_1 + z_2 + \ldots\ldots + z_n|}$$

$$= |z_1 + z_2 + \ldots\ldots + z_n| \quad [\because |\overline{z}| = |z|]$$

Square Roots of a Complex Number :

To Evaluate $\sqrt{a + i\,b}$

Solution: Let $\sqrt{a + i\,b} = x + i\,y \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$(i)

On squaring both sides of (i) , we get

$$a + ib = (x + iy)^2$$

$$\Rightarrow \quad (a + ib) = (x^2 - y^2) + i\,(2xy) \ldots\ldots\ldots\ldots\text{(ii)}$$

On comparing real parts and imaginary parts on both sides of (ii), we get

$$x^2 - y^2 = a \text{ and } 2xy = b$$

$$\Rightarrow (x^2 + y^2) = \sqrt{(x^2 + y^2)^2 + 4\,x^2\,y^2} = \sqrt{a^2 + b^2}$$

$$\therefore x^2 - y^2 = a \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(iii)}$$

And $x^2 + y^2 = \sqrt{a^2 + b^2} \ldots\ldots\ldots\ldots\ldots$(iv)

On solving (iii) and (iv), we get

$x^2 = \frac{1}{2}\left\{\sqrt{a^2 + b^2} + a\right\}$ and $y^2 = \frac{1}{2}\left\{\sqrt{a^2 + b^2} - a\right\}$

Hence, $\sqrt{a + i\,b} = \pm\left\{\sqrt{\frac{1}{2}\left\{\sqrt{a^2 + b^2} + a\right\}} + i\sqrt{\frac{1}{2}\left\{\sqrt{a^2 + b^2} - a\right\}}\right\}$

<u>Remarks</u>:Similarly , by assuming that

$\sqrt{a - i\,b}$ = (x – iy), we may find $\sqrt{a - i\,b}$.

<u>Ex</u>:Evaluate : $\sqrt{-i}$

<u>Solution</u>: Let $\sqrt{-i}$ = x – iy. . . . .(i)

On squaring both sides of (i), we get

-i = $(x – iy)^2$.

⇒ -i = $(x^2 - y^2) - i\,(2xy)$ . . . .(ii)

On comparing real parts and imaginary parts on both sides of (ii) we get

$x^2 - y^2 = 0$ and $2xy = 1$

$\therefore (x^2 + y^2) = \sqrt{(x^2 - y^2)^2 + 4x^2y^2}$

$= \sqrt{0^2 + 1^2}$

$= \sqrt{0 + 1}$

$= \sqrt{1}$

$= 1$

Thus $x^2 - y^2 = 0$ . . . . (iii) and $(x^2 + y^2) = 1$ . . . . (iv) on solving (iii) and (iv), we get $x^2 = \frac{1}{2}$ and $y^2 = \frac{1}{2}$

$\therefore x = \pm\frac{1}{\sqrt{2}}$ and $y = \pm\frac{1}{\sqrt{2}}$

Since xy>0 , so x and y are of the same sign.

$\therefore$ ($x = \frac{1}{\sqrt{2}}$, $y = \frac{1}{\sqrt{2}}$) or $\left(x = \frac{-1}{\sqrt{2}},\ y = \frac{-1}{\sqrt{2}}\right)$

Hence $\sqrt{-\mathrm{i}} = \left(\frac{1}{\sqrt{2}} - \mathrm{i}\,.\frac{1}{\sqrt{2}}\right)$ or $\left(\frac{-1}{\sqrt{2}} + \mathrm{i}\,\frac{1}{\sqrt{2}}\right)$.

(B) Sequence and Series:

A succession of numbers arranged in a definite order according to a certain given rule is called a sequence.

The number occurring at the n^{th} place of a sequence is called its n^{th} term or the general term, to be denoted by a_n. A sequence is said to be finite or infinite according as the number of terms in it is finite or infinite respectively.

A sequence can be regarded as a function whose domain is the set of natural numbers or some subset of it of the type {1,2,3,...k}. Sometimes we use the functional notation a(n) for a_n.

Series: By adding the terms of a sequence, we get a series. Let $a_1,a_2,.....,a_n$. be a given sequence. Thus the expression

$a_1+a_2+....+a_n$

is called the series associated with the given sequence. Series are often represented in compact forms, called sigma notations. Thus the series $a_1+a_2+.....+a_n$ is abbreviated as

$$\sum_{k=1}^{n} a_k$$

Ex. 1 Write the first three terms in each of the following sequence defined by the following:

i) $a_n = 2n+5$

ii) $a_n = \frac{n-3}{4}$

Solution: i) Here $a_n = 2n+5$

Substituting n = 1,2,3, we get

$a_1 = 2(1)+5 = 7$, $a_2 = 9$, $a_3 = 11$

Therefore, the required terms are 7,9 and 11.

iii) Here $a_n = \frac{n-3}{4}$

Thus $a_1 = \frac{1-3}{4} = \frac{-1}{2}$

$a_2 = \frac{-1}{4}$

$a_3 = 0$

Hence, the first three terms are $\frac{-1}{2}$, $\frac{-1}{4}$ and 0.

Progression: Sequence following patterns are called progressions.

Arithmetic Progression (A.P.): It is a sequence in which each term except the first one differs from its preceding term by a constant.

This constant difference is called the common difference of the A.P.

In an A.P. we usually denote the first term by 'a', the common difference by 'd' and the nth term by T_n.

Ex. Show that the square log a, log $(\frac{a^2}{b})$, log $(\frac{a^3}{b})$, log $(\frac{a^4}{b})$.... forms an A.P. Find its common difference.

Solution: By symmetry, we find that

$T_n = (\frac{a^n}{b^{n-1}})$ and $T_{n-1} = \log (\frac{a^{n-1}}{b^{n-2}})$

$T_n - T_{n-1} = \log(\frac{a^n}{b^{n-1}}) - \log (\frac{a^{n-1}}{b^{n-2}})$

$= \log \{ (\frac{a^n}{b^{n-1}}) \times (\frac{a^{n-1}}{b^{n-2}}) \}$

$= \log (\frac{a}{b})$ = constant

Hence the given sequence in an A.P. with common difference $\log (\frac{a}{b})$

Ex. Show that the sequence defined by Tn = $3n^2$+2 and Tn-1 = $3(n-1)^2$+2 = $3n^2$-6n+5

- ∴ $T_n - T_{n-1} = (3n^2+2) - (3n^2-6n+5)$
- ⇨ $T_n - T_{n-1} = 6^{n-3}$

This shows (Tn-Tn-1) is not independent of n and therefore, it is not constant.

Hence, the given sequence is not an A.P.

General Term of an A.P.

Theorem: Show that the nth term of an A.P. with first term 'a' and common difference 'd' is given by T_n = a+(n-1)d

Proof – Let us consider an A.P. with first term 'a' and common difference'd'. Then, the given A.P. is a, (a+d), (a+2d),........

In this A.P., we have

First term T1 = a = (a+0xd) = a+(1-1)d
Second term T2 = a+d = a+(2-1)d
Third term T3 = a+2d = a+(3-1)d
....
....
nth term Tn = a+(n-1)d
Hence, Tn = a+(n-1)d

Some properties of an A.P.

(i) If a constant is added to each term of an A.P. then the resulting progression is an A.P.
(ii) If a constant is subtracted from each term of an A.P. then the resulting progression is an A.P.
(iii) If each term of an A.P. is multiplied by the same non-zero number then the resulting progression is an A.P.
(iv) If each term of an A.P. is divided by the same non-zero number then the resulting progression is an A.P.

Ex. Show that the progression log a, log (ab), log (ab^2),.....is an A.P. Find its general term and the 10^{th} term.

Solution: We have
log (ab) – log a = log a + log b – log a = log b
log (ab^2) – log (ab) = log a + 2 log b – log a – log b = log b

$\therefore$ $(T_2-T_1) = (T_3-T_2) = ...$ = log b = constant

So, the given progression is an A.P.

Let A be the first term and d be the common difference of this A.P.

Then A = log a and d = log b
$\therefore$ general term Tn = A + (n-1)d
= log a + (n-1) log b

10^{th} term, T_{10} = log a + (10-1) log b
= log a + 9 log b
= log a + log b^9
= log (ab^9)

Ex: find the 10^{th} common term between the arithmetic series 3+7+11+15+.....and 1+6+11+16+....

Solution: Clearly, the first common term in the two series is 11. LCM of the common difference of the two series = LCM (45) = 20

Let us now consider and A.P. in which a=11 abd d=20. Every term of this AP is common term of the two given series. So, the required 10^{th} common term = 11+(10-1)20 = 11+180 = 191
Hence, 191 is the 10^{th} common term of the two given series.

Theorem: If the nth term of a progression is a linear expression in n then show that it is an A.P.

Proof: Let the nth term of a progression be a linear expression in n
Thus T_n = an+b...............(i)
Where a and b are constants.
replacing n by (n-1) in (i), we get T_{n-1} = a(n-1) + b.................(ii)
on substituting (ii) from (i), we get

[Tn-Tn-1] = a, which is constant.
This shows that the difference between any two consecutive terms of the given progression is constant.

Hence, the given progression is an A.P.

<u>Sum of n Terms of an AP</u>: Let us consider an AP containing n terms with the first term 'a', the common difference 'd' and the last term l.

Let the sum of these n terms be S_n, thus
S_n = a+(a+d)+(a+2d)+.............+(l-2d)+(l-d)+l......+(i)
writing the above series in a reverse order, we get
S_n = l+(l-d)+(l-2d)+....................+(a+2d)+(a+d)+a..............(ii)

Adding the corresponding terms of (i) & (ii), we get
2Sn = [(a+l)+(a+l)+......+ to n terms] = n(a+l)
.
.. Sn = $\frac{n}{2}$(a+l)
But l = nth term = a+(n-1)d
.
.. Sn = $\frac{n}{2}$[a+a+(n-1)d] = $\frac{n}{2}$[2a+(n-1)d]
Hence Sn = $\frac{n}{2}$[2a+(n-1)d]

Ex: Find the sum of all two digit numbers which when divided by 4 yield 1 as remainder.

Solution: Required sum
13+17+21+25+.....+97
This is an AP in which a = 13, d= (17-13)=4 and l=97
Let the number of terms be n. Then
Tn = 97 => a+(n-1)d = 97
⇨ 13+(n-1)x4 = 97 => n=22
∴ required sum = $\frac{n}{2}$ (a+l) = $\frac{22}{2}$x(13+97) = (11x110)= 1210
Hence the required sum is 1210.

Ex. The difference between any two consecutive interior angles of a polygon is 5°. If the smallest angle is 120°, find the number of sides of the polygon.

Solution: Let the number of sides of the polygon be n. Then the sum of its interior angles
= (2n-4) right angles.
= (n-2)x180^0

Since the difference between any two consecutive interior angles of the polygon is constant, its angle are in AP.

In this AP, we have a=120 and d=5

How Sn = (n-2)x180

- ⇨ $\frac{n}{2}$ {2x120+(n-1)x5} = (n-2)x180
- ⇨ $\frac{235n}{2}+\frac{5n^2}{2}$ = 180n-360 =>$5n^2$-125n+720=0
- ⇨ n^2-25n+144=0
- ⇨ $(n-9)$(n-16)=0
- ⇨ $n=9$ or n=16

But when n=16, we have

last angle = $\{120+(16-1)x5\}^0=195^0$ which is not possible.

∴ n=9

Hence the number of sides of the given polygon = 9

Arithmetic Mean: If a, A, b are in AP then we say that A is the arithmetic mean (AM) between a and b.

Insertion of a single Arithmetic Mean between a and b: Let a and b be two given numbers and let A be the arithmetic mean between a and b. Then

a, A, b are in AP

- ⇨ A-a = b-A
- ⇨ 2A = a+b
- ⇨ A = $\frac{a+b}{2}$

Hence, the arithmetic mean between a and b is $\frac{a+b}{2}$

Insertion of n Arithmetic Means between a and b: Let a and b be two given numbers and let A_1, A_2, ...A_n, b the n arithmetic means between a and b. Then a, A_1, A_2, ...A_n, b are in AP.

In this AP, we have first term = a, last term = b, number of terms = (n+2)

Let the common difference be d, then

b = T_{n+2} => b = a + (n+2-1)d

⇨ d = $\frac{b-a}{n+1}$

∴ A_1 = (a+d) = (a+$\frac{b-a}{n+1}$)

A_2 = (a+2d) = {a+$\frac{2(b-a)}{n+1}$}

.

.

A_n= (a+nd) = {a+$\frac{n(b-a)}{n+1}$}

These are the required n arithmetic means between a and b.

Ex. If $(\frac{a^n+b^n}{a^{n-1}+b^{n-1}})$ is the AM between a and b then find the value of n.

Solution: We know that the AM between a and b is $\frac{a+b}{2}$,

$\therefore$ $(\frac{a^n+b^n}{a^{n-1}+b^{n-1}})$ is the AM between a and b.

⇨ $\frac{a^n+b^n}{a^{n-1}+b^{n-1}} = \frac{a+b}{2}$

⇨ $2a^n+2b^n = a^n+a^{n-1}.b+b^{n-1}.a+b^n$

⇨ $a^n+b^n=a^{n-1}.b+b^{n-1}.a$

⇨ $a^n-a^{n-1}.b=b^{n-1}.a-b^n$

⇨ $a^{n-1}(a-b)=b^{n-1}(a-b)$

⇨ $a^{n-1}=b^{n-1}$

⇨ $(\frac{a}{b})^{n-1}=1=(\frac{a}{b})^0$

⇨ n-1=0

⇨ n=1

Hence, the required value of n is 1.

Ex: If n arithmetic means are inserted between tow numbers, prove that the sum of the means equidistant from the beginning and the end is constant.

Solution: Let $A_1, A_2, \ldots\ldots A_n$ be nAMs between a and b. Then, a, $A_1, A_2, \ldots\ldots A_n$,b are in AP

$\therefore$ mth mean from the beginning +mth mean from the end = (m+1)th term from the beginning + (m+1)th term from the end.

= {a+(m+1-1)d} + {b-(m+1-1)d} = (a+b) = constant

Hence, the sum of the means equidistant from the beginning and the end is constant.

Ex: If a,b,c are in AP, show that

(i) $\frac{1}{bc}, \frac{1}{ac}, \frac{1}{ab}$ are in AP

(ii) $a(\frac{1}{b}+\frac{1}{c}), b(\frac{1}{c}+\frac{1}{a}), c(\frac{1}{a}+\frac{1}{b})$ are in AP.

Solution:

(i) Since a,b,c are in AP, the terms obtained by dividing each term of this AP by abc are also in AP.

Consequently $\frac{1}{bc}, \frac{1}{ac}, \frac{1}{ab}$ are in AP.

(ii) a, b, c are in AP

⇨ $\frac{1}{bc}, \frac{1}{ac}, \frac{1}{ab}$ are in AP (dividing each term by abc)

⇨ $(\frac{ab+bc+ca}{bc}), (\frac{ab+bc+ca}{ca}), (\frac{ab+bc+ca}{ab})$ are in AP

⇨ $\left(\frac{ab+bc+ca}{bc}-1\right), \left(\frac{ab+bc+ca}{ca}-1\right), \left(\frac{ab+bc+ca}{ab}-1\right)$ are in AP

⇨ $\left(\frac{ab+ca}{bc}\right), \left(\frac{ab+bc}{ca}\right), \left(\frac{bc+ca}{ab}\right)$ are in AP

⇨ $a\left(\frac{1}{b}+\frac{1}{c}\right), b\left(\frac{1}{c}+\frac{1}{a}\right), c\left(\frac{1}{a}+\frac{1}{b}\right)$ are in AP

Ex: If $\frac{b+c-a}{a}, \frac{c+a-b}{b}, \frac{a+b-c}{c}$ are in AP, prove that $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in AP

Solution: $\frac{b+c-a}{a}, \frac{c+a-b}{b}, \frac{a+b-c}{c}$ are in AP

⇨ $\left(\frac{b+c-a}{a}+2\right), \left(\frac{c+a-b}{b}+2\right), \left(\frac{a+b-c}{c}+2\right)$ are in AP

⇨ $\left(\frac{a+b+c}{a}\right), \left(\frac{a+b+c}{b}\right), \left(\frac{a+b+c}{c}\right)$ are in AP

⇨ $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in AP

Ex: If a,b,c are in AP, prove that $(a-c)^2 = 4(b^2-ac)$

Solution: Since a,b,c are in AP, we have $b = \frac{1}{2}(a+c)$

RHS = $4(b^2-ac) = 4\{\frac{(a+c)^2}{4}-ac\}$

$= (a+c)^2-4ac$

$= (a-c)^2$

= LHS

Hence $(a-c)^2 = 4(b^2-ac)$

Geometrical Progression (GP): A sequence $a_1, a_2, \ldots\ldots a_n$ is called a geometrical progression, if each term is non zero and $\frac{a_{k-1}}{a_k} = r$ (constant) for all $k \geq 1$

The constant ratio is called its common ratio. In GP we usually denote the first term by a, the common ratio by r and the n^{th} term by T_n

Geometric Series: If $a_1, a_2, \ldots\ldots a_n, \ldots\ldots$ is a GP then the sum $a_1 + a_2 + \ldots\ldots + a_n + \ldots\ldots$ is called a geometric series.

General Term of a GP: Let us consider a GP in which first term = a and common ration = r.

Then the GP is a, ar, ar^2,.........................

Its first term, $T_1 = a = a\text{-}r^{(1-1)}$

Second term, $T_2 = ar = a\text{-}r^{(2-1)}$

....

....

∴ n^{th} term, $T_n = ar^{(n-1)}$

Hence $T_n = ar^{(n-1)}$

Ex: If a, b, c are three consecutive terms of an AP and x, y, z are three consecutive terms of a GP, then prove that

$x^{b-c} . y^{c-a} . z^{a-b} = 1$

Solution: It is given that a, b, c are in AP

So 2b = a+c and x,y,z are in GP, so $y = \sqrt{xz}$

$\therefore \quad x^{b-c} . y^{c-a} . z^{a-b}$

$= x^{b-c} . (\sqrt{xz})^{c-a} . z^{a-b}$

$= x^{b-c} . x^{\frac{c-a}{2}} . z^{\frac{c-a}{2}} . z^{a-b}$

$= x^{b-c+\frac{c-a}{2}} . z^{\frac{c-a}{2}+a-b}$

$= x^{\frac{2b-c-a}{2}} . z^{\frac{c+a-2b}{2}}$

$= x^{\frac{a+c-c-a}{2}} . y^{\frac{c+a-c-a}{2}}$

$= x^0 . y^0$

$= 1$

Ex: If a, b, c, d are in GP, prove that

$(a^2+b^2+c^2)(b^2+c^2+d^2) = (ab+bc+cd)^2$

<u>Solution</u>: Let r be the common ratio of the GP a, b, c, d

Then $b = ar$, $c = ar^2$ and $d = ar^3$

$\therefore$ LHS $= (a^2+b^2+c^2)(b^2+c^2+d^2)$

$= (a^2+a^2r^2+a^2r^4)(a^2r^2+a^2r^4+a^2r^6)$

$= a^4r^2(1+r^2+r^4)^2$

And RHS $= (ab+bc+cd)^2$

$= (a^2r+a^2r^3+a^2r^5)^2$

$= a^4r^2(1+r^2+r^4)^2$

Hence, $(a^2+b^2+c^2)(b^2+c^2+d^2) = (ab+bc+cd)^2$

Ex: If a, b, c are in AP; b, c, d are in GP and $\frac{1}{c}, \frac{1}{d}, \frac{1}{e}$ are in AP, then show that a, c, e are in GP.

<u>Solution</u>: a, b, c are in AP => 2b = a+c..................(i)

b, c, d are in GP => $c^2 = bd$.....................(ii)

$\frac{1}{c}, \frac{1}{d}, \frac{1}{e}$ are in AP => $\frac{2}{d} = [\frac{1}{c}+\frac{1}{e}] = \frac{(c+e)}{ce}$..........(iii)

Now, $c^2 = bd \Rightarrow c^2 = \frac{(a+c)}{2} \cdot \frac{2ce}{(c+e)}$

$\Rightarrow c = \frac{(a+c)e}{c+e}$

$\Rightarrow c\,(c+e) = (a+c)e$

$\Rightarrow c^2 = ae$

=>a, c, e are in AP

Hence a, c, e are in AP

Ex. Find all sequences which are simultaneously AP and GP.

<u>Solution</u>: Let $a_1, a_2, \ldots\ldots\ldots\ldots, a_n, \ldots\ldots\ldots\ldots$ be a sequence which is both an AP as well as a GP.

Consider three consecutive terms a_n, a_{n+1}, a_{n+2} of this AP and GP.

Then $2_{an+1} = a_n + a_{n+2}$............(i)

And, if r be the common ratio of the GP, then

$a_n = ar^{n-1}$, $a_{n+1} = a_1r^n$ and $a_{n+2} = a_1r^{n+1}$

Putting these values in (i), we get

$2\,a_1r^n = a_1r^{n-1} + a_1r^{n+1}$

- ⇨ $r^2 - 2r + 1 = 0$
- ⇨ $(r-1)^2 = 0$
- ⇨ $r = 1$

Putting r = 1, we get $a_n = a_1$, $a_{n+1} = a_1$ and $a_{n+2} = a_1$

Thus, we get a constant sequence $a_1, a_2, \ldots\ldots\ldots\ldots\ldots\ldots$

Hence, a constant sequence is the only sequence which is both AP and GP.

<u>n^{th} Term from the end of a GP</u>: Let a be the first term, r be the common ratio and l be the last term of a given GP. Then,

2^{nd} term from the end = $\frac{l}{r} = \frac{l}{r^{(2-1)}}$

3^{rd} term from the end = $\frac{l}{r^2} = \frac{l}{r^{(3-1)}}$

....

....

n^{th} term from the end = $\frac{l}{r^{(n-1)}}$

Hence, the n^{th} term from the end = $\frac{l}{r^{(n-1)}}$

For solving problems on GP, it is always convenient to take:

(i) 3 numbers in GP as $\frac{a}{r}$, ar, ar

(ii) 4 numbers in GP as $\frac{a}{r^3}, \frac{a}{r}$, ar, ar^3

(iii) 5 numbers in GP as $\frac{a}{r^2}, \frac{a}{r}$, a, ar, ar^2

(iv) the terms as a, ar, ar^2,.....................when their product is not given

Ex. If the product of three numbers in GP is 216 and the sum of their products in pairs is 156, find the numbers.

<u>Solution</u>: Let the required numbers be $\frac{a}{r}$, a, ar

Then $\frac{a}{r}$ x a x ar = 216

⇨ $a^3 = 216 = 6^3$

⇨ a = 6

and $\frac{a}{r}$ x a + a x ar + $\frac{a}{r}$ x ar = 156

⇨ $a^2 (\frac{1}{r} + r + 1) = 156$

⇨ $6^2 (1+r+r^2) = 156r$

⇨ $36 (r^2+r+1) = 156r$

⇨ $3 (r^2+r+1) = 13r$

⇨ $3r^2-10r+3 = 0$

⇨ $(3r-1) (r-3) = 0$

⇨ $r = \frac{1}{3}$ or $r = 3$

So, the required numbers are 18, 6, 2 or 2, 6, 18.

<u>Sum of n Terms of a GP</u>: Let us consider a GP with the first term a and the common ratio r.

Then $S_n = a+ar+................+ar^{n-1}$..............(i)

<u>Case 1</u>: If r = 1, we have

$S_n = a+a+.........+$ to n terms = na

<u>Case 2</u>: If r ≠ 1, we have

$rS_n = ar+ar^2+..............+ar^{n-1}+ar^n$..................(ii) on subtracting (ii) from (i), we get

$(1-r)S_n = (a-ar^n) = a(1-r^n)$

⇨ $S_n = \frac{a(1-r^n)}{(1-r)}$ or $S_n = \frac{a(r^n-1)}{(r-1)}$

∴ $S_n = \frac{a(1-r^n)}{(1-r)}$, when $r < 1$

$\frac{a(r^n-1)}{(r-1)}$, when $r>1$

Ex: Find the sum of the sequence

7,77, 777, 7777, ... to n terms

<u>Solution</u>: This is not a GP, however, we can relate it to a GP by writing the terms as

S_n = 7+ 77 + 777 + 7777 + ...+ to n terms`

$= \frac{7}{9}$[9 + 99+...................+ to n terms]

$= \frac{7}{9}$ [(10-1) + (10^2-1) +.............+ to n terms]

$= \frac{7}{9}$ [(10+10^2+.....+n terms) – (1+1+1+.......+ n terms)]

$= \frac{7}{9}\left[\frac{10(10^n-1)}{10-1} - n\right]$

$= \frac{7}{9}\left[\frac{10(10^n-1)}{9} - n\right]$

Ex: A person has 2 parents, 4 grandparents, 8 great grandparents and so on. Find the number of his ancestors during the ten generations preceding his own.

<u>Solution</u>: Here a = 2, r = 2 and n = 10

Using the sum formula

$S_n = \frac{a(r^n-1)}{(r-1)}$

We have $S_{10} = 2\,(2^{10}-1) = 2046$

Hence, the number of ancestors preceding the person is 2046.

Geometric Mean(GM): Let a and b be two given numbers. We say that G is the geometric mean (GM) between a and b if a, G, b are in GP.

If a, G, b are in GP

⇨ $\frac{G}{a} = \frac{b}{G}$

⇨ $G^2 = ab$

⇨ $G = \sqrt{ab}$

Remarks:

(i) The GM between two positive numbers is positive.

(ii) The GM between two negative number is negative.

(iii) The GM between two numbers of opposite signs does not exist.

Ex: Find two positive numbers a and b whose AM and GM are 34 and 16 respectively.

Solution: We have

$\frac{a+b}{2} = 34$ and $\sqrt{ab} = 16$

⇨ a+b = 68 and ab = 256

⇨ $(a-b) = \sqrt{(a+b)^2 - 4ab} = \sqrt{(68)^2 - 4x256} = \sqrt{3600} = 260$

⇨ a+b = 68 and a-b = 260

⇨ (a = 64, b = 4) or (a = 4, b = 64)

Hence the required numbers are (a = 64, b = 4) or (a = 4, b = 64)

N Geometric Means between two given Numbers:

Let a and b be two given numbers, we say that $G_1, G_2, \ldots G_n$ are n geometric means between a and b, if a $G_1, G_2, \ldots G_n$, b are in G.P.

This GP contains (n+2) terms.

Let the common ratio of this GP be r. Then

$T_{n+2} = b \Rightarrow ar^{n+1} = b \Rightarrow r = (\frac{b}{a})^{\frac{1}{n+1}}$

$\therefore G_1 = ar = a \,.\, (\frac{b}{a})^{\frac{1}{n+1}}$

$G_2 = ar^2 = a \,.\, (\frac{b}{a})^{\frac{2}{n+1}}$

$G_3 = ar^3 = a \cdot (\frac{b}{a})^{\frac{3}{n+1}}$

....

....

$G_n = ar^n = a \cdot (\frac{b}{a})^{\frac{n}{n+1}}$

Ex: Insert three numbers between 1 and 256 so that the resulting sequence is a G.P.

Solution: Let G_1, G_2, G_3 be three numbers between 1 and 256 such that 1, G_1, G_2, G_3, 256 is a G.P.

Therefore $256 = 1xr^4 \Rightarrow r = \pm 4$

for r = 4, we have $G_1 = ar = 4$, $G_2 = ar^2 = 16$ $G_3 = ar^3 = 64$

Similarly, for r = -4, numbers are -4, 16 and -64.

Hence, we can insert 4, 16, 64 between 1 and 256 so that the resulting sequences are in GP.

Relationship between A.M. and G.M.:

Let A and G be A.M and G.M of two given positive real numbers a and b respectively.

Thus $A = \frac{a+b}{2}$ and $G = \sqrt{ab}$

Thus we have $A\text{-}G = \frac{a+b}{2} - \sqrt{ab} = \frac{a+b-2\sqrt{ab}}{2}$

$= \frac{(\sqrt{a}-\sqrt{b})^2}{2} >, 0$

⇨ $A \geq G$

Hence A.M. ≥ G.M.

Ex: Find the value of n so that $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ may be the geometric mean between a and b, where $a \neq b$

Solution: $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ is GM between a and

⇨ $\frac{a^{n+1}+b^{n+1}}{a^n+b^n} = b = a^{\frac{1}{2}} b^{\frac{1}{2}}$

⇨ $[a^{n+1} + b^{n+1}] = (a^n + b^n)\, a^{\frac{1}{2}} b^{\frac{1}{2}}$

⇨ $[a^{n+1} + b^{n+1}] = a^{n+\frac{1}{2}} b^{\frac{1}{2}} + a^{\frac{1}{2}} b^{n+\frac{1}{2}}$

⇨ $(a^{n+1}$- $a^{n+\frac{1}{2}}\, b^{\frac{1}{2}})$ = $(b^{n+\frac{1}{2}} a^{\frac{1}{2}}$ - $b^{n+\frac{1}{2}})$

⇨

$\Longrightarrow a^{(n+\frac{1}{2})}$, $(a^{\frac{1}{2}}$ - $b^{\frac{1}{2}})$ = $b^{(n+\frac{1}{2})}$ $(a^{\frac{1}{2}}$ - $b^{\frac{1}{2}})$

$\Longrightarrow a^{(n+\frac{1}{2})}$ = $b^{(n+\frac{1}{2})}$[∵ $(a^{\frac{1}{2}}$- $b^{\frac{1}{2}})$ ≠ 0 , since a ≠ b]

$\Longrightarrow (\frac{a}{b})^{(n+\frac{1}{2})}$ = 1 = $(\frac{a}{b})^0$

$\Longrightarrow$ n + $\frac{1}{2}$ = 0 $\Longrightarrow$ n = - $\frac{1}{2}$

Hence n = - $\frac{1}{2}$.

Ex : If a, b, c, d are four distinct positive numbers in GP then prove that a+d>b+c.

Solution : a, b, c, d are in GP

$\Longrightarrow$a, b, c are in GP

∵ AM > GM

∴$\frac{a+c}{2}$> b $\Longrightarrow$ a + c > 2b- - - - - - - - - - - - - - - - -(i)

Again a, b, c, d are in GP

$\Longrightarrow$b, c, d are in GP

∴$\frac{b+d}{2}$> c $\Longrightarrow$ b + d >2c - - - - - - - - - - - - - - - -(ii)

From (i) and (ii), we get

a + c + b + d > 2b + 2c

Hence a + d > b + c.

Ex: If $x \in R$, find the minimum value of $3^x + 3^{(1-x)}$.

Solution : we know that

AM > GM

∴for all $x \in R$, we have

$\frac{3^x + 3^{(1-x)}}{2} > \sqrt{3^x x\, 3^{(1-x)}}$

$\Longrightarrow \frac{3^x + 3^{(1-x)}}{2} > \sqrt{3}$

$\Longrightarrow 3^x + 3^{(1-x)} > \sqrt[2]{3}$.

Hence the minimum value of $3^x + 3^{(1-x)}$ is $\sqrt[2]{3}$ for any $x \in R$.

Infinite Geometric series: Let us consider an infinite GP with first term a and common ratio r, where $-1 < r < 1$, i.e., $|r| < 1$

The sum of n terms of a GP is given by

$Sn = \frac{a\,(1-r^2)}{1-r} \Longrightarrow Sn = \left\{\frac{a}{(1-r)} \frac{a\,r^n}{(1-r)}\right\}$------------(i)

Since $|r| < |$, so when n increases then r^n decreases thus when $n \longrightarrow \infty$ then $r^n \longrightarrow 0$

∴ $\text{Lim } n \Rightarrow \infty \frac{r^n}{(1-r)} = 0$

Hence, the sum of one an infinite G.P. is given by

$$\therefore \lim_{n\to\infty} Sn = \lim_{n\to\infty} \left\{\frac{a}{(1-r)} - \frac{ar^n}{(1-r)}\right\}$$

$$= \frac{a}{(1-r)} - \lim_{n\to\infty} \frac{ar^n}{(1-r)}$$

$$= \left\{\frac{a}{(1-r)} - 0\right\}$$

$$= \frac{a}{(1-r)}$$

This gives, S = $\frac{a}{(1-r)}$, when |r| <|

If r≥, | then sum of an infinite GP is S= ∞ .

Ex: Use geometric series to express 0.555------------- = 0. $\bar{5}$as a rational number.

Solution : we have

$$0.\bar{5} = 0.555\text{-------------}$$
$$= 0.5 + 0.05 + 0.005 + \_\_\_\_\_\_\_\_\_ + \infty.$$
$$= \frac{5}{10} + \frac{5}{10^2} + \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \infty.$$
$$= \frac{5/10}{(1-\frac{1}{10}} = \frac{5}{9}$$

Hence $0.\bar{5} = \frac{5}{9}$.

Sum of n terms of special series:

Sum of first n natural Numbers :

Prove that (1+ 2 + 3 +_ _ _ _ + n) = $\frac{1}{2}$ n (n+1)

$\sum n = \frac{1}{2}$ n (n+1)

Proof : Consider the series (1+ 2 + 3 +_ _ _ _ _ _ _+n). this is an arithmetic series in which a= 1, d= 1 and l = n.

∴ Sn = $\frac{n}{2}$ (1+n)_ _ _ _ _ _ _ _ _ _ [∵ Sn = $\frac{1}{2}$ (a+l)] or $\sum n = \frac{1}{2}$n (n+1).

Sum of the square of first n Natural Numbers:

Prove that $(1^2 + 2^2 + 3^2 + \_\_\_\_\_\_\_ + n^2) = \frac{1}{6} n (n+1) (2n+1)$

Proof – Let $S_n = 1^2 + 2^2 + \_\_\_\_\_\_\_ + n^2$

Consider the identity $K^3 - (K-1)^3 = 3K^2 - 3K + 1$ _ _ _ _ _ _ _ _ _(i)

Putting K = 1, 2, 3,_ _ _ _ _ _ (n-1), n successively in (i), we get

$1^3 - 0^3 = 3 \cdot 1^2 - 3.1 + 1$

$2^3 - 13 = 3.2^2 - 3.2 + 1$

$3^3 - 2^3 = 3.3^2 - 3.3 + 1$

_ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _

_ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _

$(n-1)^3 - (n-2)^3 = 3.(n-1)^2 - 3(n-1) + 1$

$n^3 - (n-1)^3 = 3.n^2 - 3.n + 1$

Adding columnwise, we get

$n^3 - 0^3 = 3(1^2 + 2^2 + \_\_\_\_\_\_\_\_ + n^2) - 3(1+2+3+\_\_\_ + n) + n$

$\Rightarrow n^3 = 3\sum n^2 - 3\,\frac{1}{2} n (n+1) + n$

$\Rightarrow 3\sum n^2 = n^3 + \frac{3}{2} n (n+1) - n$

$\Rightarrow 3\sum n^2 = \frac{1}{2}(2n^3 + 3n^2 + n)$

$\Rightarrow \sum n^2 = \frac{1}{6}(2n^3 + 3n^2 + n)$

$\Rightarrow \sum n^2 = \frac{1}{6} n (2n^2 + 3n + 1) = \frac{1}{6} n (n+1) (2n+1)$

Hence $\sum n^2 = \frac{1}{6} n (n+1) (2n+1)$

Sum of the cubes of first n natural numbers:

Prove that $(1^3 + 2^3 + \_\_\_\_\_ + n^3) = \left\{\frac{1}{2} n (n+1)\right\}^2$

i.e. $\sum n^3 = \left\{\frac{1}{2} n (n+1)\right\}^2$

proof : let $S_n = 1^3 + 2^3 + \_\_\_\_\_\_\_\_ + n^3$

Consider the identity $K^4 - (K-1)^4 = 4K^3 - 6K^2 + 4K - 1$

Putting K = 1, 2, 3, _ _ _ _ , (n-1) , n successively , we get

$1^4 - 0^4 = 4.1^3 - 6.1^2 + 4.1 - 1$

$2^4 - 1^4 = 4.2^3 - 6.2^2 + 4.2 - 1$

_ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _

_ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _

$(n-1)^4 - (n-2)^4 = 4.(n-1)^3 - 6.(n-1)^2 + 4(n-1) - 1$

$n^4 - (n-1)^4 = 4.n^3 - 6.n^2 + 4.n - 1$

Adding column wise, we get

$(n^4 - 0^4) = 4.(1^3 + 2^3 + \_\_\_\_\_ + n^3) - 6.(1^2 + 2^2 + \_\_\_\_\_\_\_\_ + n^2)$

$+ 4(1+2+\_\_\_ + n) - (1 + 1 + \_\_\_\_\_ \text{to n terms})$

$\Rightarrow n^4 = 4\sum n^3 - 6\sum n^2 + 4\sum n - n$

$\Rightarrow n^4 = 4\sum n^3 - 6 . \frac{1}{6} n (n+1)(2n+1) + 4\frac{1}{2} n (n+1) - n$

$\Rightarrow 4\sum n^3 = n^4 + n(n+1)(2n+1) - 2n(n+1) + n$

$\Rightarrow 4\sum n^3 = n^4 + 2n^3 + 3n^2 + n - 2n^2 - 2n + n$

$\Rightarrow 4\sum n^3 = n^4 + 2n^3 + n^2$

$\Rightarrow \sum n^3 = \frac{1}{4} n^2 (n^2 + 2n + 1)$

$\Rightarrow \sum n^3 = \frac{1}{4} n^2 (n+1)^2$

$\Rightarrow \sum n^3 = \left\{\frac{1}{2} n (n+1)\right\}^2$

$\Rightarrow \sum n^3 = (\sum n)^2$

Ex: Find the sum $1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) +$_ _ _ _ _to n terms.

We have $T_k = (1^2 + 2^2 +$_ _ _ _$+K^2)$

$= \left\{\frac{1}{6} K (K+1) (2K+1)\right\}$

$= \frac{1}{6} (2K^3 + 3K^2 + K)$

$\therefore$ Sn $= \sum_{K=1}^{n} T_K$

$= \frac{1}{6} . \sum_{K=1}^{n} (2K^3 + 3K^2 + K)$

$= \frac{1}{6} . 2 \sum_{K=1}^{n} K^3 + \frac{1}{6} .3 \sum_{K=1}^{n} K^2 + \frac{1}{6} . \sum_{K=1}^{n} K$

$= \frac{1}{3} \sum_{K=1}^{n} K^3 + \frac{1}{2} . \sum_{K=1}^{n} K^2 + \frac{1}{6} \sum_{K=1}^{n} K$

$= \frac{1}{3} . \{\frac{1}{2} n (n+1)\}^2 + \frac{1}{2} . \frac{1}{6} n (n+1) (2n+1) + \frac{1}{6} . \frac{1}{2} n (n+1)$

$= \frac{1}{12} n^2 (n+1)^2 + \frac{1}{12} n (n+1)(2n+1) + \frac{1}{12} n (n+1)$

$= \frac{1}{12} n (n+1) \left[n (n+1) + (2n+1) + 1\right]$

$= \frac{1}{12} n (n+1)^2 (n+2)$

Ex: find the sum of n terms of the series

$$\frac{1}{(2\ x\ 5)} + \frac{1}{(5\ x\ 8)} + \frac{1}{(8\ x\ 11)} + ---------$$

Solution: we have

$$T_K = \frac{1}{(Kth\ \ term\ of\ 2,5,8,.....)\ x\ \ Kth\ \ term\ of\ 5,8,11,....)}$$

$$T_K = \frac{1}{\{\ 2+(K-1)_3\}\ x\ \{5+(K-1)3\}}$$

$$= \frac{1}{(3K-1)(3K+2)} = \frac{1}{3}\left\{\frac{1}{(3K-1)} - \frac{1}{(3K+2)}\right\}$$

$$\therefore\ T_K = \frac{1}{3}\left\{\frac{1}{3K-1} - \frac{1}{3K+2}\right\}$$

Putting K = 1, 2, 3, ………,n successively , we get

$$T_1 = \frac{1}{3}\left(\frac{1}{2} - \frac{1}{5}\right)$$

$$T_2 = \frac{1}{3}\left(\frac{1}{5} - \frac{1}{8}\right)$$

$$T_n = \frac{1}{3}\left\{\frac{1}{3n-1} - \frac{1}{3n+2}\right\}$$

Adding columnwise, we get

Sn = $(T_1 + T_2 + ... + T_n)$

$$= \frac{1}{3}\left(\frac{1}{2} - \frac{1}{3n+2}\right)$$

Sn = $\frac{n}{2\ (3n+2)}$

(C) PERMUTATION AND COMBINATIONS

The Factorial: The continued product of first n natural numbers is called the 'n factorial' and is denoted by n!

i.e., n! = 1x2x3x.....x (n-1)xn

Thus 3! = 1x2x3=6

Clearly, n! is defined for positive integers only.

Zero Factorial: It does not make any sense to define it as the product of the integers from 1 to zero. So, we define 0! = 1

Note: Factorial of proper fractions or negative integers are not defined.

Deduction: n! = 1x2x3x......(n-1)xn

= [1x2x3x....x(n-1)]xn

= (n-1)!xn

n! = n(n-1)!

Ex. 1. Convert the following products into factorials:

i) 6.7.8.9.10
ii) 2.4.6.8-10

Solution:

i) $6.7.8.9.10 = \dfrac{1x2x3x4x5x6x7x8x9x10}{1x2x3x4x5}$

$= \dfrac{10!}{5!}$

(ii) 2.4.6.8.10 = (2x1)x(2x2)x(2x3)x(2x4)x(2x5)

$= 2^5x1x2x3x4x5$

$=2^5x5!$

Ex.2 Find the LCM of 4!, 5! and 6!

Solution: We have 5! = 5x4!

6! = 6x5x4!

∴ LCM of 4!, 5!, 6! = LCM [4!, 5x4!, 6x5x4!]
= 4! x 5 x 6
= 6!
= 720

Ex.3 Prove that (n!+1) is not divisible by any natural number between 2 and n.

Solution: Let m is divisible by K and r be any natural number between 1 and k. If m+ris divided by k, then we obtain r as the remainder.
Since n! = 1.2.3.......(n-1).n it follows that n! is divisible by every natural number between 2 and n. So, (n!+1), when divided by any natural number between 2 and n, leaves 1 as the reminder. Hence, (n!+1) is not divisible by any natural number between 2 and n.

Fundamental Principles of counting:

Fundamental Principle of multiplication – if there are two jobs such that one of them can be completed in m ways, and when it has been completed in any one of these m ways, second job can be completed in n ways, then the two jobs in succession can be completed in mxn ways.

Ex. In a class there are 10 boys and 8 girls. The teacher wants to select a boy and a girl to represent the class in a function. In how many ways can the teacher make this selection?

Solution: Here the teacher is to perform two jobs:
(i) Selecting a boy among 10 boys, and
(ii) Selecting a girl among 8 girls.

The first of these can be performed in 10 ways and the second in 8 ways. Therefore by the fundamental principle of multiplication, the required number of ways is 10x8=80.

Fundamental Principle of Addition: If there are two jobs such that they can be performed independently in m and n ways respectively, then either of two jobs can be performed in (m+n) ways.

Ex. In a class there are 10 boys and 8 girls. The teacher wants to select either a boy or a girl to represent the class in a function. In how many ways the teacher can make this selection?

Solution: Here the teacher is to perform either of the following two jobs.

i) Selecting a boy among 10 boys.
ii) Selecting a girl among 8 girls.

The first of these can be performed in 10 ways and the second in 8 ways. Therefore by fundamental principle of addition either of the two jobs can be performed in (10+8) = 18 ways. Hence the teacher can make the selection of either a boy or a girl in 18 ways.

Ex. In a monthly test, the teacher decides that there will be three questions, one from each of Exercises 7, 8 and 9 of the text book. If there are 12 questions in Exercise 7, 8 in and 9 of the text book. If there are 12 questions in Exercise 7, 18 in Exercise 8 and 9 in Exercise 9, in how many ways can three questions be selected?
Solution: There are 12 questions in exercise 7, so one question from exercise 7 can be selected in 12 ways, Exercise 8 contains 18 questions, so second question can be selected in 18 ways. There 9 questions in exercise 9. So, third question can be selected in 9 ways. Hence, three questions can be selected in 12x18x9=1944 ways.

Ex. How many three digit numbers can be formed without using the digits 0,2,3,4,5 and 6?

Solution: We have to determine the total number of three digit numbers formed by using the digits 1,7,8,9. Clearly the repetition of digits is allowed.

A three digit number has three places i.e., units's, ten's and hundred's. Unit's place can be filled by any of the digits 1,7,8,9. So unit's place can be filled in 4 ways. Similarly each one of the ten's and hundred's place can be filled in 4 ways.

Total number of required numbers = 4x4x4=64

Ex. How many numbers are there between 100 and 1000 in which all the digits are distinct?

Solution: A number between 100 and 1000 has three digits. So, we have to form all possible 3 digit numbers with distinct digits. We cannot have 0 at the hundred's place. So, the hundred's place can be filled with any of the 9 digits 1,2,.....,9. So, there are 9 ways of filling the hundred's place.

Now 9 digits are left including 0. So tens place can be filled with any of the remaining 9 digits in 9 ways. Now the unit's place can be filled with in any of the remaining 8 digits. So there are 8 ways of filling the unit's place.

Hence, the total number of required numbers = 9x9x8 = 648.

Ex. How many numbers are there between 100 and 1000 such that 7 is in the unit's place.

Solution: Every number between 100 and 1000 is a three digit number. So, we have to form 3 digit numbers with 7 at the unit's place by using the digits 0,1,2,....,9. Clearly repetitions of digits is allowed. The hundred's place can be filled with any of the digits from 1 to 9. So, hundred's place can be filled in 9 ways.

Now the ten's place can be filled with any of the digits from 0 to 9. So ten's place can be filled in 10 ways. Since all the numbers have digit 7 at the unit's place, so unit's place can be filled in only one way.

Hence, by the fundamental principle of counting the total number of numbers between 100 and 1000 having 7 at the unit's place = 9x10x1 = 90.

Ex. How many numbers are there between 100 and 1000 such that at least one of their digit is 7.

Solution: Clearly a number between 100 and 1000 has 3 digits.

$\therefore$ Total number of 3 digit numbers having at least one of their digit as 7.

= (Total number of three digit numbers) – (Total number of 3 digit numbers in which 7 does not appear at all)

Total number of 3 digit numbers - we have to form three digit numbers by using the digits 0,1,2,3,........,9.

Clearly, hundred's place can be filled in 9 ways and each of the ten's and units place can be filled in 10 ways.

So, the total number of 3 digit numbers = 9x10x10 = 900

Total number of 3 digit number in which 7 does not appear at all. Here we have to form three digit numbers by using the digits 0 to 9 except 7.

So hundred's place can be filled in 8 ways and each of the ten's and one's place can be filled in 9 ways. So total number of three digit numbers in which 7 does not appear at all = 8x9x9.

Hence total number of 3 digit numbers having at least one of their digits as 7

= 9x10x10x-8x9x9
= 252

Ex. How many numbers are there between 100 and 1000 which have exactly one of their digits as 7?

Solution: A number between 100 and 1000 contains 3 digits. So, we have to form 3 digit numbers having exactly one of their digits as 7. Such type of numbers can be divided into three types.

i) Those three digit numbers that have 7 in the unit's place but not in any other place.

   Thus there are 8x9x1= 72

ii) Those three digit numbers that have 7 in the ten's place but not in any other place.
   Thus there are 8x1x9 = 72

iii) Those three digit numbers that have 7 in the hundred's place but not at any other place.
   So, there are 1x9x9 = 81

Hence, the total number of required type of numbers = 72+72+81 = 225

Ex. A gentleman has 6 friends to invite. In how many ways can he send invitation cards to them, if he has three servants to carry the cards?

Solution: Since a card can be sent by any one of the three servants, so the number of ways of sending the invitation card to the first friend = 3. Similarly invitation cards can be sent to each of the six friends in 3 ways.

So, the required number of ways = 3x3x3x3x3x3 = 3^6 = 729

Ex. How many three digit odd numbers can be formed by using the digits 1,2,3,4,5,6 if:

i) The repetition of digits is not allowed?
ii) The repetition of digits is allowed?

Solution: For a number to be odd, we may have 1,3 or 5 at the units place. So, there are 3 ways of filling the unit's place.

i) Since the repetition of digits is not allowed, the ten's place can be filled with any of the remaining 5 digits in 5 ways. Now, four digits are left. So hundred's place can be filled in 4 ways.

So, the required of number = 3x5x4 = 60

ii) Since the repetition of digits is allowed, so each of the ten's and hundred's place can be filled in 6 ways.

Hence, the required number of numbers
=3x6x6 = 108

Ex. In how many ways 5 rings of different type can be worn in 4 fingers?

Solution: The first ring can be worn in any of the 4 fingers. So, there are 4 ways of wearing it. Similarly, each one of the other rings can be worn in 4 ways.

Hence, the required number of ways

= 4x4x4x4x4= 4^5

Ex. By using the digits 0,1,2,3,4 and 5 (repetitions not allowed) numbers are formed by using any number of digits. Find the total number of non-zero numbers that can be formed?

Solution: Number of numbers = Number of digit number + No. of 2 digit numbers +....No. of 6 digit numbers
= 5+5x5+5x5x4+....+5x5x4x3x2x1 = 5+25+100+300+600+600 = 1630

PERMUTATIONS: Each of the arrangements which can be made by taking same or all of a number of things is called a permutation.

Ex. If there are three objects, then the permutations of these objects, taking two at a time, are ab, ba, bc, cb, ac, ca. So the number of permutations of three different things taken two at a time is 6.

A NOTATION: If n and r are positive integers such that 1≤r≤n, then the number of all permutations of n distint things taken r at a time is denoted by the symbol P (n,r) or $^{n}\mathrm{Pr}$
Thus $^{n}\mathrm{Pr}$ or P(n,r) = Total number of permutations of n distinct things, taken r at a time.

Theorem: Let r and n be positive integers such that 1≤r≤n. Then the number of all permutations of n distint things taken r at a time is given by P(n,r) = $^{n}\mathrm{Pr}$ = n(n-1) (n-2)......{n-(r-1)}

Theorem: Prove that P(n,r) = $^{n}\mathrm{Pr} = \frac{n!}{(n-r)!}$

Proof:

$^{n}\mathrm{Pr}$ = n(n-1) (n-2)....(n-(r-1))

$= \frac{\mathrm{n(n-1)(n-2)....(n-(r-1))(n-r)(n-(r+1))....3.2.1}}{\mathrm{(n-r)(n-(r+1))....3.2.1}}$

$= \frac{n!}{(n-r)!}$

Theorem: The number of all permutations of n distinct things, taken all at a time is n!

Proof: P(n,n) = n(n-1) (n-2)....(n-(n-1))
= n(n-1) (n-2)....3.2.1
=n!

Theorem: Prove 0! = 1

Proof: We have

$^{n}\mathrm{Pr} = \frac{n!}{(n-r)!}$

=> $^{n}\mathrm{Pr} = \frac{n!}{(n-n)!}$

=>n! = $\frac{n!}{0!}$

=> 0! = $\frac{n!}{n!}$

=> 0! = 1

Ex. In how many ways three different rings can be worn in four fingers with at most one in each finger?

Solution: The total number of ways is some as the number of arrangements of 4 fingers, taken 3 at a time. So required number of ways

$^{4}P_{3} = \frac{4!}{(4-3)!} = \frac{4!}{1!} = 4! = 24$

ALITER: Let R1, R2, R3 be three rings. Since R1 can be put in any one of the four fingers. So there are four ways in which R1 can be worn. Now R2, can be worn any one of the remaining three fingers in 3 ways. In the remaining 2 fingers R3 can be worn in 2 ways. So by the fundamental principle of counting the total number of ways in which three different rings can be worn in four fingers = 4x3x2= 24

Ex. Seven athletes are participating in a race. In how many ways can the first three prizes be worn?

Solution: The total number of ways in which first three prizes can be won is the number of arrangements of seven different things taken 3 at a time.

So the required number of ways = $\frac{7!}{(7-3)!}$

= $\frac{7!}{4!}$ = 7x6x5 = 210

ALTER: First prize can be won in seven ways. Second prize can be won by any one of the remaining six athlets in 6 ways. Five athelets are left. So third prize can be won by any one of the remaining 5 athelts in 5 ways. Hence by the fundamental principal of counting, the required number of ways = 7x6x5 = 210.

Ex. In how many ways can 6 persons stand in a queue?

Solution: The number of ways in which 6 persons can stand in a queue is same as the number of arrangements of 6 different things taken all at a time.

Hence the required number of ways = ${}^6P_6$ = 6! = 720

Ex. It is required to seat 5 men and 4 women in a row so that the women occupy the even places. How many such arrangements are possible?

Solution: In all 9 persons are to be seated in a row and in the row of 9 positions there are exactly four even places viz second, fourth, sixth and eighth. It is given that these four even places are to be occupied by 4 women. This can be done ${}^4P_4$ ways. The remaining 5 positions can be filled by the 5 men in ${}^5P_5$ ways. So by the fundamental principle of counting, the number of seating arrangements as required is

${}^4P_4 \times {}^5P_5$ = 4! x 5! = 24x120 = 2880

Ex. How many different signals can be given using any number of flags from 5 flags of different colours?

Solution: The signals can be made by using at a time one or two or three or four or five flags.

Hence by the fundamental principle of the total number of signals = ${}^5P_1+{}^5P_2+{}^5P_3+{}^5P_4+{}^5P_5$ = 5+5x4+5x4x3+5x4x3x2+5x4x3x2x1
= 5+20+60+120+120
= 325

Ex. How many numbers lying between 100 and 1000 can be formed with the digits 1,2,3,4,5 if the repetition of digits is not allowed?

Solution: Every number lying between 100 and 1000 is a three digit number. Therefore, we have to find the number of permutations of five digits 1,2,3,4,5 taken three digit at a time.

Hence, the required number of numbers

= ${}^5P_3 = \frac{5!}{(5-3)!} = \frac{5!}{2!}$ = 5x4x3 = 60

Ex. How many four digit numbers are there with distinct digits?

Solution: The total number of arrangements of ten digits 0,1,2,3,4,5,6,7,8,9 taking 4 at a time is ${}^{10}P_4$. But these arrangements also include those numbers which have 0 at thousand's place. Such numbers are not four digit numbers.

When 0 is fixed at thousand's place, we have to arrange remaining 9 digits by taking 3 at a time. The number of such arrangements is ${}^9P_3$.

Hence the total number of four digit numbers = ${}^{10}P_4 - {}^9P_3$ = 5040 – 504 = 4536.

Ex. The different letters of an alphabet are given words with five letters are formed from these given letters. Determine the number of words which have alleast one letter repeated.

Solution: The number of 5 letters words which can be formed from 10 letters when one or more of its letters is repeated = 10x10x10x10x10 = 10^5

The number of 5 letter words which can be formed when none of their letters is repeated

= Number of arrangements of 10 letters by taking 5 at a time.
= ${}^{10}P_5$ = 30240
Hence the number of 5 letters words which have at least one of their letter repeated = 10^5 – 30240 = 69760
Ex. In how many ways 7 pictures can be hung from 5 pictures nails on a wall?

Solution: The number of ways in which 7 pictures can be hung from 5 picture nails on a wall is same as the number of arrangements of 7 things, taking 5 at a time.

Hence the required number = ${}^7P_5$

$= \frac{7!}{(7-5)!} = \frac{7!}{2!}$ = 7x6x5x4x3 = 2520

Ex. There are six periods in each working day of a school. In how many ways can one arrange 5 subjects such that each subject is allowed at least one period?

Solution: Five subjects can be arranged in 6 periods in ${}^6P_5$ ways. Now, one period is left and it can by allowed to any one of the five subjects. So, number of ways in which remaining one period can be arranged is 5.

Hence, the total number of arrangements = ${}^6P_5$ x 5 = 3600.

Permutation under certain conditions:

Theorem: Prove that the number of all permutations of n different objects taken r at a time when a particular object is to be always included in each arrangement is
r. ${}^{n-1}P_{r-1}$

Theorem 2: Prove that the number of permutations of n distinct objects taken r at a time, when a particular object is never taken in each arrangement, is ${}^{n-1}P_r$

Theorem 3: Prove that the number of permutations of n different objects taken r at a time in which two specified objects always occur together is 2! (r-1) ${}^{n-2}P_{r-2}$.

Ex1 In how many ways can the letters of the word PENCIL, be arranged so that

i) N is always next to E
ii) N and E are always together.

Solution:

i) Let us keep EN together and consider it as one letter. Now, we have 5 letters which can be arranged in a row in ${}^5P_5$ = 5! = 120 ways. Hence the total number of ways in which N is always next to E is 120.

ii) Keeping E and N together and considering it as one letter, we have 5 letters which can be arranged in ${}^5P_5$ = 5! ways. But E and N can be put together 2! ways (viz EN, NE). Hence the total number of ways = 5! x 2! = 240.

Ex 2. How many different wards can be formed with the letters of the word EQUATION so that the words begin and end with a consonant?

Solution: There are 3 consonants and all words should begin and end with a consonant. So first and last place can be filled with 3 consonants in ${}^3P_2$ ways. Now, the remaining 6 places are to be filled up with the remaining 6 letters in ${}^6P_6$ ways. Hence the required number of words = ${}^3P_2 x {}^6P_6$ = 6x720 = 4320.

Ex. In how many ways 5 boys and 3 girls can be seated in a row so that no two girls are together?

Solution: The 5 boys can be seated in a row in ${}^5P_5$=5! ways. In each of these arrangements 6 places are created, shown by the cross-marks, as given below

x B x B x B x B x B x

Since no two girls are to sit together, so we may arrange 3 girls in 6 places. This can be done in ${}^6P_3$ ways i.e., 3 girls can be seated in ${}^6P_3$ ways.

Hence the total number of seating arrangements = ${}^5P_5$ x ${}^6P_3$ = 5! x 6 x 5 x 4 = 14400.

Ex. In how many ways can 9 examination papers be arranged so that the best and the worst papers are never together?

Solution: The total number of arrangements of 9 papers = ${}^9P_9$ = 9!

Considering the best and the worst papers as one paper, we have 8 papers which can be arranged in ${}^8P_8$ = 8! ways. But the best and worst papers can be put together in 2! ways. So the number of permutations in which the best and the worst papers can be put together = 2! x 8!

Hence the number of ways in which the best and the worst papers never come together = 9! – 2! x 8! = 7 x 8! = 282240.

Ex. How many four digit numbers divisible by 4 can be made with the digits 1,2,3,4,5 if the repetition of the digits is not allowed?

Solution: Recall that a number is divisible by 4 if the number formed by the last two digits is divisible by 4. The digits of unit's and ten's places can be arranged as follows-

T	H	T	0
x	x	1	2
x	x	2	4
x	x	3	2
x	x	5	2

Now, corresponding each such way the remaining three digits at thousands and hundreds places can be arranged in ${}^3P_2$ ways.

Hence the required number of numbers = ${}^3P_2$ x 4 = 3! x 4 = 24

Ex. Five boys and five girls form a line with the boys and girls alternating. Find the number of ways of making the line.

Solution: 5 boys can be arranged in a line in ${}^5P_5$ = 5! ways. Since the boys and girls are alternating, so corresponding each of the 5! ways of arrangements of 5 boys we obtain 5 places make cross as shown below:

i) B1xB2xB3xB4xB5x
ii) xB1xB2xB3xB4xB5

Clearly, 5 girls can be arranged in 5 places mark by cross in (5!+5!) ways.
Hence, the total number of ways of making the line = 5! x (5!+5!) = $2(5!)^2$.

Ex. If 20 persons were invited for a party, in how many ways can they and the host be seated at a circular table? In how many of these ways will two particular persons be seated on either side of the host?

Solution: i) Clearly there are 21 persons, including the host, to be seated round a circular table. These 21 person can be seated normal a circular table in (21-1)! = 20! ways.

Permutations of objects not all distinct:

Theorem: The number of mutually distinguishable permutations of n things, taken all at a time, of which p are alike of one kind q alike of second kind such that p+q = n, is $\frac{n!}{p!q!}$

Ex. How many different words can be formed with the letters of the word MISSISSIPPI?

Solution: There are 11 letters in the given word of which 4 are S's, 4 are I's and 2 are P's. So total number of words is the number of arrangements of 11 things, of which 4 are similar of one kind, 4 are similar of second kind and 2 are similar of third kind i.e., $\frac{11!}{4!4!2!}$
Hence, the total number of words = $\frac{11!}{4!4!2!}$ = 34650

Ex. How many different words can be formed by using all the letters of the word 'ALLAHABAD'?

i) In how many of them vowels occupy the even positions?
ii) In how many of them both L do not come together?

Solution: There are 9 letters in the word 'ALLAHABAD' out of which 4 are A's, 2 are L's and the rest are all distinct.

So the requisite number of words = $\frac{9!}{4!2!}$ = 7560

i) There are 4 vowels and all are alike i.e., 4 A's. Also there are 4 even places viz2^{nd} , 4^{th}, 6th, 8^{th}. So these 4 even places can be occupied by 4 vowels in $\frac{4!}{4!}$ = 1 way. Now, we are left with 5 places in which 5 which two are alike (2 L's) and other distinct, can be arranged in $\frac{5!}{2!}$ ways.

Hence the total number of words in which vowels occupy the even places = $\frac{5!}{2!}$ x $\frac{4!}{4!}$ = $\frac{5!}{2!}$ = 60

ii) Considering both L together and treating them as one letter we have 8 letters out of which A repeats 4 times and others are distinct. These 8 letters can be arranged in $\frac{8!}{4!}$ ways.

So the number of words in which both L come together = $\frac{8!}{4!}$ = 1680

Hence, the number of words in which both L do not come together.
= Total no. of words – No. of words in which both L come together.
= 7560-1680
= 5880

Ex. How many arrangements can be made with the letters of the word 'MATHEMATICS'. In how many of them vowels are together?

Solution: There are 11 letters in the word 'MATHEMATICS' of which two are M's, two are A's, two are T's and all other are distinct. So required number of arrangements = $\frac{11!}{2!X2!X2!}$ = 4989600

There are 4 vowels viz A, E, A, I considering these four vowels as one letter. We have 8 letters (M, T, H, M, T, C, S and one letter obtained by combining all vowels, out of which M occurs twice, T occurs twice and rest all different. These 8 letters can be arranged in $\frac{8!}{2!X2!}$ ways.

But the four vowels (A, E, A, I) can be put together in $\frac{4!}{2!}$ ways.

Hence the total number of arrangements in which vowels are always together = $\frac{8!}{2!X2!}$ x $\frac{4!}{2!}$ = 10080x12 = 120960

<u>Circular Permutations</u>: Permutation of objects in a row is called linear permutions. If we arrange the objects along a closed curve viz. a circle, the permutations are known as circular permutations. Every linear arrangement has a beginning and an end, but there is nothing like beginning or end in a circular permutation. Thus, in a circular permutation, we consider one object as fixed and the remaining objects are arranged as in case of linear arrangements.

Theorem: The number of circular permutations of n distinct objects is (n-1)!

Note: In the above theorem anti-clockwise and clockwise order of arrangements are considered as distinct permutations.

2. If anti-clockwise and clock-wise order of arrangements are not distinct e.g. arrangements of beads in a necklace, arrangements of flowers in a garland etc. thus the number of circular permutations of n distinct items is $\frac{1}{2}[(n-1)!]$

Ex. In how many ways can 8 students be seated in a (i) line (ii) circle?

Solution: (i) The number of ways in which 8 students can be seated in a line = $^8P_8 = 8! = 40320$.

(iii) The number of ways in which 8 students can be seated in a circle = $(8-1)! = 7! = 5040$.

Ex. (i) In how many ways can 5 persons be seated around a circular table?
(ii) In how many of these arrangements will two particular persons be next to each other?

Solution: (i) F persons can be seated around a circular table in $(5-1)! = 4! = 24$ ways.
(ii) Considering two particular persons as one person we have 4 persons in all. These 4 persons can be seated around a circular table in $(4-1)! = 3!$ ways. But 2 particular persons can be arranged among themselves in $2!$ ways.
Hence the total number of arrangements = $3! \times 2! = 12$
(ii) Let P1, P2 be two particular persons and H be the host. These two particular person can be seated on either side of the host in the following two ways:

(i) P_1HP_2
(ii) P2HP1

Consider the two particular person and the host as one person, we have 18 persons in all. These 19 persons can be seated round a circular table in $(19-1)! = 18!$ ways. But two particular person can be seated on either side of the host in 2 ways. So, the number of ways of seating 21 person at a circular table with two particular persons on either side of the host = $18! \times 2$

Ex. Find the number of ways in which 8 different flowers can be strung to form a garland so that 4 particular flowers are never separated.

Solution: Considering 4 particular flowers as one flower, we have five flowers which can be strung to form a garland in $4!$ ways. But, 4 particular flowers can be arranged in $4!$ ways. Then the required number of ways = $4! \times 4! = 576$.

Ex. Find the number of ways in which 10 different beads can be arranged to form a necklace.

Solution: Ten different beads can be arranged in circular forms in $(10-1)! = 9!$ ways. Since there is no distinction between the clockwise and anti-clockwise arrangements. So, the required number of arrangements = $\frac{1}{2}9!$

CENTRAL BOARD OF SECONDARY EDUCATION

STUDY MATERIAL: APPLIED MATHEMATICS

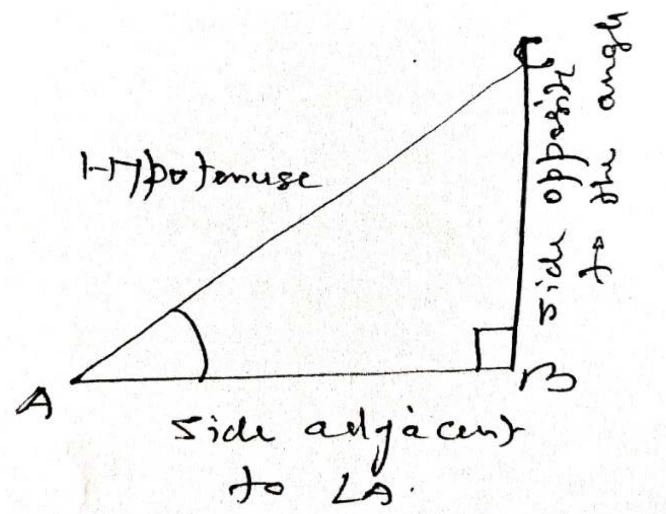
{Class XI}

UNIT 8: TRIGONOMETRY

UNIT 8:

TRIGONOMETRY

Trigonometric Ratios:



Let us take a right triangle ABC. Here ∠CAB is an acute angle. BC is the side opposite to angle A. AC is the hypotenuse of the right triangle and the side AB is a part of ∠A. So, we call it the side adjacent to angle A

Note that the position of sides change when we consider angle C in place of A.

The trigonometric ratios of the angle A in right triangle ABC are defined as follows:

$$\text{Sine A} = \frac{BC}{AC}$$

$$\text{Cos A} = \frac{AB}{AC}$$

$$\text{Tan A} = \frac{BC}{AB}$$

$$\text{Cosec A} = \frac{AC}{BC}$$

$$\text{Sec A} = \frac{AC}{AB}$$

$$\text{Cot A} = \frac{AB}{BC}$$

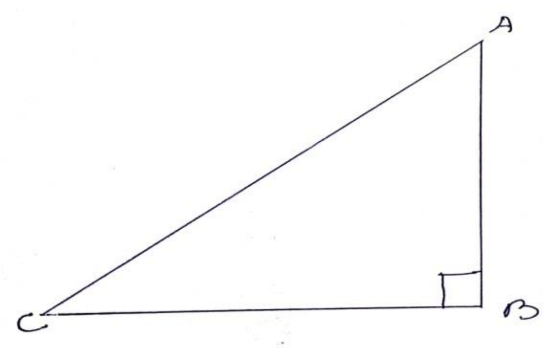
Note that the ratios Cosec A, Sec A and Cot A are respectively the reciprocals of the ratios Sin A, Cos A and Tan A

Also, observe that Tan A = $\frac{BC}{AB} = \frac{BC/AC}{AB/AC} = \frac{Sin\ A}{Cos\ A}$

and Cot A = $\frac{Cos\ A}{Sin\ A}$

So, the trigonometric ratios of an acute angle in a right triangle express the relationship between the angle and the length of its sides. It is now clear that the values of the trigonometric ratios of an angle do not vary with the lengths of the side of the triangle, if the angle remains the same.

Trigonometric Identities:



An equation involving trigonometric ratios of an angle is called a trigonometric identity, if it is true for all values of the angle involved.

In Δ ABC, right angled at B, we have

$$AB^2+BC^2 = AC^2 \ldots\ldots\ldots\ldots\text{(i)}$$

Dividing each term of (i) by AC^2, we get

$$\frac{AB^2}{AC^2}+\frac{BC^2}{AC^2} = \frac{AC^2}{AC^2}$$

$$\text{i.e; } \left(\frac{AB}{AC}\right)^2+\left(\frac{BC}{AC}\right)^2 = \left(\frac{AC}{AC}\right)^2$$

$$\text{i.e; } (\text{Cos A})^2+(\text{Sin A})^2 = 1$$

$$\text{Cos}^2\text{A}+\text{Sin}^2\text{A} = 1 \ldots\ldots\ldots\ldots\text{(ii)}$$

This is true for all A such that $0^0 \leq A \leq 90^0$

So, this is a trigonometric identity

Let us now divide (i) by AB^2, we get

$$\frac{AB^2}{AB^2}+\frac{BC^2}{AB^2} = \frac{AC^2}{AB^2}$$

$$\text{or } \left(\frac{AB}{AB}\right)^2+\left(\frac{BC}{AB}\right)^2 = \left(\frac{AC}{AB}\right)^2$$

i.e, $1+\tan^2 A = \text{Sec}^2 A$…………..(iii)

So, (iii) is true for all A such that $0^0 \leq A \leq 90^0$

Again dividing by (i) by BC^2, we get

$$\frac{AB^2}{BC^2}+\frac{BC^2}{BC^2} = \frac{AC^2}{BC^2}$$

$$\text{i.e, } \left(\frac{AB}{BC}\right)^2+\left(\frac{BC}{BC}\right)^2 = \left(\frac{AC}{BC}\right)^2$$

i.e, $\text{Cot}^2 A+1 = \text{Cosec}^2 A$………..(iv)

Note that Cosec A and Cot A are not defined for $A = 0^0$. Therefore (iv) is true for all A such that $0^0 < A \leq 90^0$

Using these identities, we can express each trigonometric ratio in terms of other trigonometric ratios i.e, if any one of the ratios is known, we can also determine the values of other trigonometric ratios.

Ex. Express the ratios CosA, TanA and SecA in terms of SinA.

Solution: Since $\text{Cos}^2 A+\text{Sin}^2 A = 1$, therefore

$\text{Cos}^2 A = 1-\text{Sin}^2 A$

i.e, $\text{CosA} = \pm\sqrt{1-\text{Sin}^2\text{A}}$

This gives $\text{CosA} = \sqrt{1-\text{Sin}^2\text{A}}$

Hence $\text{TanA} = \frac{SinA}{CosA} = \frac{SinA}{\sqrt{1-\text{Sin}^2\text{A}}}$

And $\text{SecA} = \frac{1}{CosA} = \frac{1}{\sqrt{1-\text{Sin}^2\text{A}}}$

Ex. Prove that SecA (1-SinA) (SecA+TanA) = 1

Solution: LHS = SecA (1-SinA) (SecA+TanA)

$= \frac{1}{CosA}$ (1-SinA) ($\frac{1}{CosA}$+ $\frac{SinA}{CosA}$)

$= \frac{(1-\text{SinA})(1+\text{SinA})}{\text{Cos}^2\text{A}}$

$= \frac{1-\text{Sin}^2\text{A}}{\text{Cos}^2\text{A}}$

$= \frac{\text{Cos}^2\text{A}}{\text{Cos}^2\text{A}}$

= 1

= RHS

Ex: $\sqrt{\frac{1+SinA}{1-SinA}}$ = SecA+TanA

LHS = $\sqrt{\frac{1+SinA}{1-SinA}}$

$= \sqrt{\frac{(1+SinA)(1+SinA)}{(1-SinA)(1+SinA)}}$

$= \sqrt{\frac{(1+SinA)^2}{1-Sin^2A}}$

$= \sqrt{\frac{(1+SinA)^2}{Cos^2A}}$

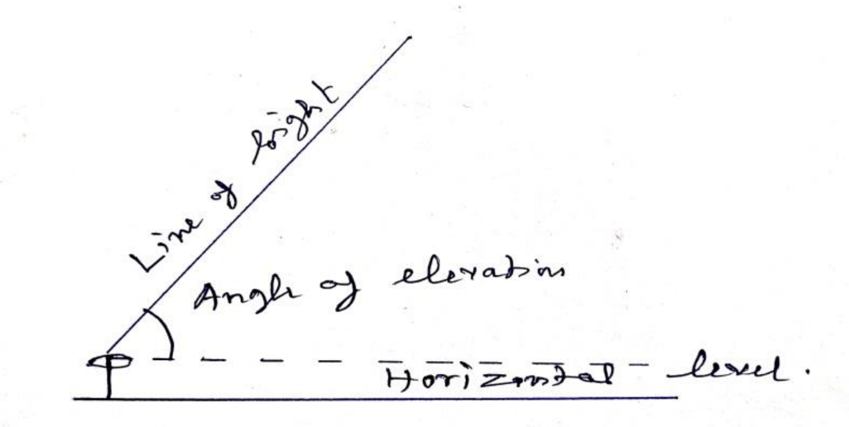
$= \frac{1+SinA}{CosA}$

$= \frac{1}{CosA} + \frac{SinA}{CosA}$

= SecA + TanA

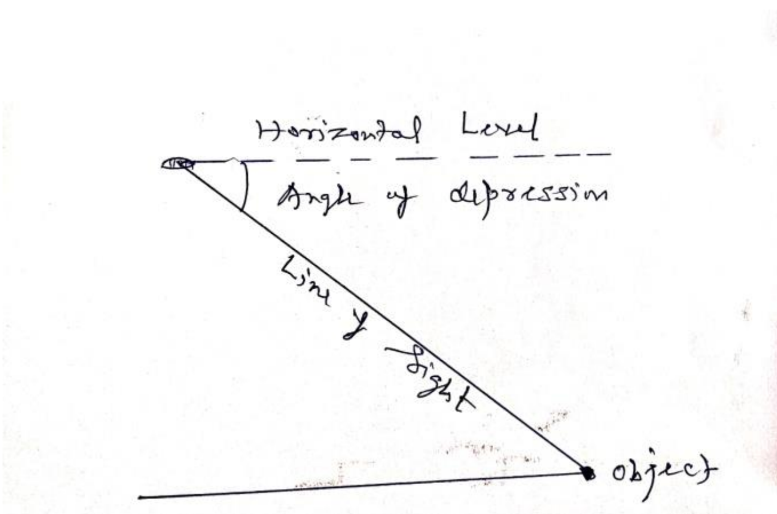
= RHS

Heights and Distances:



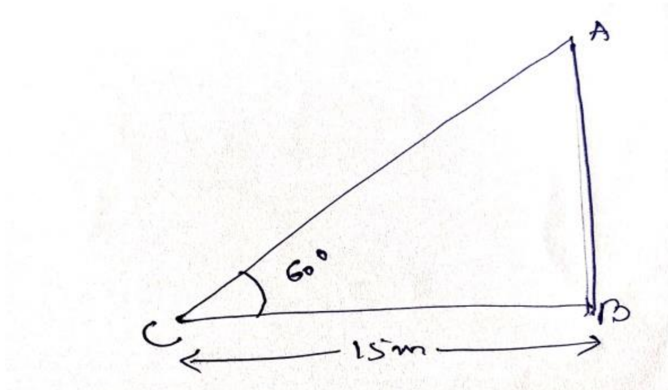
The line of sight is the line drawn from the eye of an observer to the point in the object viewed by the observer.

The angle of elevation of the point viewed is the angle formed by the line of sight with the horizontal when the point being viewed is above the horizontal level i.e, the case when we raise our head to look at the object.



The angle of depression of a point on the object being viewed is the angle formed by the line of sight with the horizontal when the point is below the horizontal level i.e, the case when we lower our head to look at the point being viewed.

Ex. A tower stands vertically on the ground. From a point on the ground, which is 15m away from the foot on the tower, the angle of elevations of the top of the tower is found to be 60^0. Find the height of the tower.



First let us draw a simple diagrams to represent the problem. Here AB represents the tower, CB is the distance of the point from the tower and ∠ACB is the angle of elevation. We need to determine the height of the tower i.e, AB. Also, ACB is a triangle, right-angled at B. To solve the problem, we choose the trigonometric ratio Tan 60^0, at the ratio involves AB and BC.

Now Tan $60^0 = \frac{AB}{BC}$

i.e, $\sqrt{3} = \frac{AB}{15}$

i.e, AB = $15\sqrt{3}$

Hence, the height of the tower is $15\sqrt{3}$ m.